

## Exercises 11

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

## 1 Key Exercises

**Question 1** Consider the three points

$$\mathbf{A} = [-1 : 1 : 0], \quad \mathbf{B} = [2 : 2 : 4], \quad \mathbf{C} = [3 : 1 : 4].$$

- (1) Show these points are collinear,
- (2) For each affine chart, find a point at infinity that is also collinear to these three points.

**Question 2** For each of the following set of points, check whether they are collinear, and if so calculate their cross-ratio  $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ .

- (1)  $\mathbf{A} = [1 : 0 : -2]$ ,  $\mathbf{B} = [-2 : 1 : -1]$ ,  $\mathbf{C} = [-4 : 1 : 3]$ ,  $\mathbf{D} = [5 : -2 : 0]$
- (2)  $\mathbf{A} = [0 : 1 : -1]$ ,  $\mathbf{B} = [-1 : 2 : 1]$ ,  $\mathbf{C} = [1 : 1 : -4]$ ,  $\mathbf{D} = [1 : 0 : 2]$
- (3)  $\mathbf{A} = [1 : 0 : -1]$ ,  $\mathbf{B} = [0 : 1 : 1]$ ,  $\mathbf{C} = [1 : 1 : 0]$ ,  $\mathbf{D} = [1 : 5 : 4]$

**Question 3** For each trio of points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$ , find a fourth point  $\mathbf{D}$  such that  $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = 3$ .

- (1)  $\mathbf{A} = [1 : 3 : 2]$ ,  $\mathbf{B} = [1 : 1 : 1]$ ,  $\mathbf{C} = [1 : -1 : 0]$
- (2)  $\mathbf{A} = [0 : 1 : 2]$ ,  $\mathbf{B} = [1 : 0 : 2]$ ,  $\mathbf{C} = [-1 : 1 : 0]$

**Question 4** Consider the conic sections

$$C = \left\{ (u, v) \in \mathbb{A}^2 \mid \frac{u^2}{4} + \frac{v^2}{9} = 1 \right\}, \quad D = \left\{ (u, v) \in \mathbb{A}^2 \mid u^2 - 4v^2 = 1 \right\}.$$

Calculate a projective transformation  $T$  that transforms  $C$  to  $D$  on the third affine chart.

## 2 Extra Exercises

**Question 5** Let  $T: \mathbb{P}^2 \rightarrow \mathbb{P}^2$  be a projective transformation that sends

$$T([0 : 1 : 2]) = \mathbf{A}, \quad T([0 : 2 : 1]) = \mathbf{B}, \quad T([0 : 0 : 1]) = \mathbf{C}.$$

For each set of points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$ , find the point  $\mathbf{X} \in \mathbb{P}^2$  such that  $T(\mathbf{X}) = \mathbf{D}$ .

(1)  $\mathbf{A} = [2 : 3 : 0]$ ,  $\mathbf{B} = [1 : 3 : 0]$ ,  $\mathbf{C} = [1 : 1 : 0]$ ,  $\mathbf{D} = [1 : 2 : 0]$

(2)  $\mathbf{A} = [2 : 1 : 4]$ ,  $\mathbf{B} = [2 : 1 : 1]$ ,  $\mathbf{C} = [0 : 0 : 1]$ ,  $\mathbf{D} = [2 : 1 : 0]$

**Question 6** Find a condition on the ordering of four collinear points  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  such that  $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) < 0$ .