Exercises 10

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 For each of the following projective transformations, write down the induced transformation on each affine chart. State whether the induced transformation on each chart is affine or rational.

(1) $T\colon \mathbb{P}^1 \to \mathbb{P}^1, \quad T([x:y]) = [5x : 2x+3y]$

(2) $T\colon \mathbb{P}^2 \to \mathbb{P}^2, \quad T([x:y:z]) = [x+y : 2y : y+z]$

Question 2 For each of the following affine/rational transformations, find a projective transformation that induces the transformation on some affine chart.

(1) $f(u,v) = (3v, 2u+1)$

(2) $f(u,v) = \left(\dfrac{3v}{2u+v+1}, \dfrac{u+v}{2u+v+1}\right)$

2 Extra Exercises

Question 3 Let (x, y) be a Cartesian coordinate system for $\mathbb{A}^2$. Let C be the curve with implicit equation $x^2 - y^2 = 1$. Let D be the curve with equation $2xy = 1$.

(1) Sketch the two curves C and D.

(2) Find (with proof) an affine transformation that maps C to D.

(3) Deduce that D is a hyperbola.

Question 4 Let (x, y) be a Cartesian coordinate system for $\mathbb{A}^2$ and let C be the curve with implicit equation $y = ax^2 + bx + c$ for $a, b, c \in \mathbb{R}$ and $a \neq 0$. By considering the affine transformation f with matrix

$$\begin{bmatrix} 0 & \frac{1}{a} & \frac{b^2-4ac}{4a^2} \\ 1 & 0 & \frac{b}{2a} \\ 0 & 0 & 1 \end{bmatrix}$$

prove that C is a parabola.

Question 5 In the course we have shown how to write $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^0$, where $\mathbb{A}^0$ is a point at infinity.

Show that one can write $\mathbb{P}^n$ as

$$\mathbb{P}^n = \bigcup_{i=0}^{n} \mathbb{A}^i.$$

Question 6 Let $T\colon \mathbb{P}^n \to \mathbb{P}^n$ be a projective transformation such that the induced transformation on each affine chart is an affine transformation. Show that T is of the form

$$T([x_1 : \cdots : x_{n+1}]) = [\lambda_1 x_1 : \cdots : \lambda_{n+1} x_{n+1}], \quad \lambda_i \in \mathbb{R} \setminus \{0\}.$$