Exercises 1

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Consider the following vectors in $\mathbb{R}^2$:

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}, \mathbf{a} = \begin{bmatrix}2\\3\end{bmatrix}, \mathbf{b} = \begin{bmatrix}3\\0\end{bmatrix},$$

(1) Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$.

(2) Show that $\{\mathbf{a}, \mathbf{b}\}$ is a basis for $\mathbb{R}^2$.

(3) Show that $\{\mathbf{e}_1, \mathbf{b}\}$ is *not* a basis for $\mathbb{R}^2$.

Question 2 State whether each of the following maps $\langle -, - \rangle$ define an inner product on $\mathbb{R}^3$. [*where* $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T}, \mathbf{y} = (y_1, y_2, y_3)^\mathsf{T}$.]

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$

(2) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3$

(3) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 + x_2 y_1 + x_3 y_3$

Question 3 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an (ordered) basis of $\mathbb{E}^3$. Consider the ordered set of vectors $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ defined by $\mathcal{B}$ via:

(1) $\mathbf{f}_1 = \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1$, $\mathbf{f}_3 = \mathbf{e}_3$

(2) $\mathbf{f}_1 = \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{f}_3 = \mathbf{e}_3$

(3) $\mathbf{f}_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{f}_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{f}_3 = \mathbf{e}_3$

(4) $\mathbf{f}_1 = \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1$, $\mathbf{f}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ where $\lambda \in \mathbb{R}$ is an arbitrary coefficient.

For each set of vectors, write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. Is $\mathcal{C}$ a basis?

2 Extra exercises

Question 4 Consider the sets of polynomials

$$V = \left\{ax^2 + bx + c \ \mid \ a, b, c \in \mathbb{R}\right\}, T = \left\{x^2 + px + q \ \mid \ p, q \in \mathbb{R}\right\}$$

with the natural operations of addition and multiplication of polynomials. [*You may assume these operations satisfy commutativity, associativity and distributivity.*]

(1) Which of these are vector spaces (over $\mathbb{R}$), and why?

(2) Show the polynomials $1, x, x^2$ are linearly independent in V.

(3) Calculate the dimension of V.

Question 5 Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be vectors of a vector space V. Show that if at least one of the vectors is equal to $\mathbf{0}$, then they are linearly dependent.