Applied Calculus 1

1013 M

by Aram Dermenjian

February 29, 2020

Week 0

Welcome to Applied Calculus I at York University! This year we will be going through a ton of various stuff in order to get you better equipped with the tools of calculus. Use these notes to help guide you through the course. Keep them with you everywhere you go, be diligent in your work and you'll be happy with your results at the end.

How to use these notes: These notes are different than most math class notes. They have holes and gaps in them. They are missing information! These are the notes I will be using in class (literally) and so it should be easier for you to follow along. Rather than writing everything, you only need to fill in the gaps. If you need to add extra notes to help you understand, use the margins! These are YOUR notes. Use them wisely.

It's recommended that you at least quickly glance over the notes before class starts (just 2-5 minutes before lecture starts should be ok).

For those of you with the "official" textbook, note that I will include what section in the book we are doing for each day so you can keep track. Note that we have 36 class and around 425 pages to run through. That's roughly 12 pages per class! Don't fall behind, it's gonna be an exciting race.

Good luck, I believe in you.

Week 1

6–10 Jan 2020

1 Day 1: 6 Jan 2020

Like any first class, we go over the basics today.

My info:

- Email
- Dahdaleh 2028

Office Hours:

- M 10.30-13
- W 10.30-12
- F 10.30-13
- Appointment

Grading:

- Quizzes: 10% - Mondays. Top 10
- Midterm 1: 30% - Second-chance
- Midterm 2: 30% - Missed exams
- Final: 30% - deferred

Other:

- dermenjian.com/moodle
- Dates on syllabus

What is calculus good for? Calculus is used throughout the world for many things. Most people know that calculus is used in physics, chemistry and economics, but it's also used in construction, medicine, geology. If you plan on working in any of the sciences, then you'll likely be using calculus.

1.1 Functions - §1.1 – 1.3

A function is nothing more than a rule where you give it something and get back something else. As an example:

More explicitly, a *function* f is a rule that assigns to each x in a set D to *exactly* one element $f(x)$, in a set E. This is normally denoted $f : D \to E$ where D is called the *domain* and E is called the *codomain*. The *range* of f is the set of all possible values of $f(x)$ for every x in D.

Exercise 1.1 Create your own function that does not use any numbers nor atoms! When finished, compare with your neighbor.

2 Day 2: 8 Jan 2020

(1) Language - English/French are both acceptible when handing in assignments.

(2) Quizzes - Only 7 of 9 will be counted

(3) 2nd chance exams - Will take higher of the two scores, but max is now 75%.

(4) If ask what's on exam/quiz - I'll make it harder. Everything is on exam.

1.1 Recall:

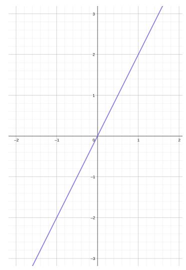
(1) Calculus is used everywhere!

(2) $f : D \to E$, Domain, codomain, range

Example 1.2 Let's look at some examples.

(1) A *linear function* is a function whose graph is a line (*e.g.* functions of the form $f(x) = mx + b$. Let f be the linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x$.

Domain: $\mathbb{R}$ Codomain: $\mathbb{R}$ Range: $\mathbb{R}$

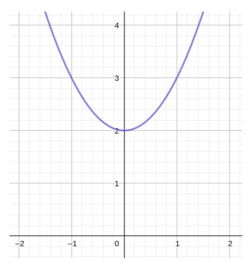


(2) A *polynomial* f is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$$

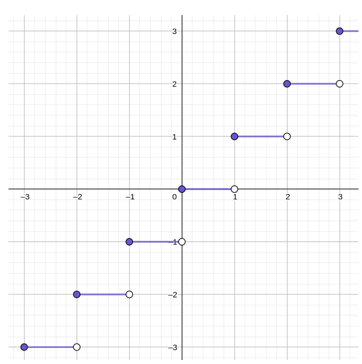
where $a_0, \ldots, a_n$ are known as the *coefficients* and, if $a_n \neq 0$, then f is said to have *degree* n.
Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial of degree 2 such that $f(x) = \underline{x^2 + 2}$.

Domain: $\mathbb{R}$ Codomain: $\mathbb{R}$ Range: $\mathbb{R}_{\geq 2}$



(3) Let f be the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \underline{\lfloor x \rfloor}$.

Domain: $\mathbb{R}$ Codomain: $\mathbb{R}$ Range: $\mathbb{Z}$



(4) A *rational function* f is a ratio of two polynomials:

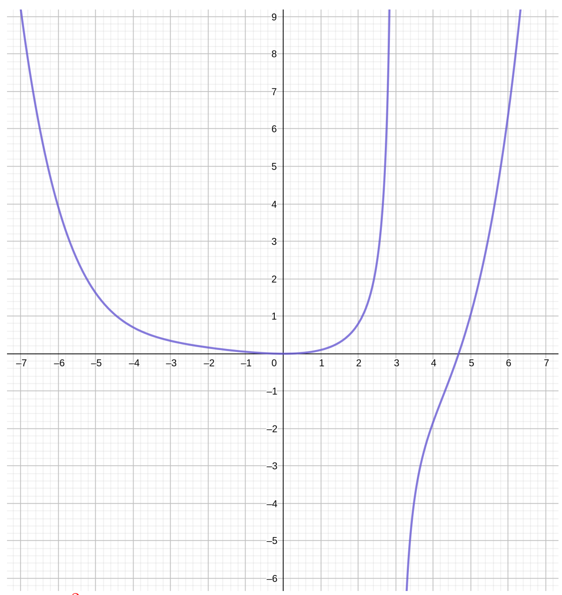
$$f(x) = \frac{P(x)}{Q(x)}.$$

Let f be the rational function

$$f(x) = \underline{\frac{x^7 - x^5 - 7x^4 + x^3 - 2000x^2 + 67}{10000(x-3)}}$$

Domain: $\underline{\mathbb{R}\setminus\{3\}}$ Range: $\underline{\mathbb{R}}$

Why am I not asking for the codomain? *Didn't give it to you.*

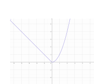


Explain why x and $\frac{x^2}{x}$ are not the same function even though we can divide. (Domains are different)

(5) *Piecewise defined function:* is a function which has different formulas in different parts of their domains. Let f be the piecewise defined function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ |x| & \text{if } x < 0 \end{cases}$$

Domain: $\underline{\mathbb{R}}$ Codomain: $\underline{\mathbb{R}}$ Range: $\underline{\mathbb{R}^+}$



An *even* function is a function f that satisfies $f(-x) = f(x)$. An *odd* function is a function f that satisfies $f(-x) = -f(x)$.

Example 1.3 Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^3 + x^7$. Is it even or odd? odd

Go through previous examples and see which ones are even/odd. Point out the graphs! We can flip over diagonal to see if it's odd, and over y to see if it's even.

Example 1.4 Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \sin(x)$ and $g : \mathbb{R} \to \mathbb{R}$ be the function $g(x) = \cos(x)$. Which is even and which is odd?

Even: cos Odd: sin

Give these two problems at the same time and give them roughly 5 minutes to do it

Exercise 1.5 With a person next to you: Give an example of a function that is neither even nor odd: $x^2 + x$

Exercise 1.6 With a person next to you: Give an example of a function that is both even and odd: 0

Recall that $[a, b]$ is a *closed interval* of $\mathbb{R}$ and (a, b) is an *open interval* of $\mathbb{R}$. A function f is called *increasing* on an interval I if:

$$f(x) < f(y) \text{ for every } x < y \text{ in } I.$$

A function f is called *decreasing* on an interval I if:

$$f(x) > f(y) \text{ for every } x < y \text{ in } I.$$

Let f and g be two functions. Then

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example 1.7 Let $f(x) = x^2$ and $g(x) = x - 1$.

$$(f+g)(x) = x^2 + x - 1 \qquad (f-g)(x) = x^2 - x + 1$$
$$(fg)(x) = x^3 - x^2 \qquad \frac{f}{g}(x) = \frac{x^2}{x-1}$$

The *composition* of two functions is the *composite function* $f \circ g$ where

$$f \circ g(x) = f(g(x))$$

The order matters for composition, just like subtraction and division!

Example 1.8 Let $f(x) = x^2$ and $g(x) = x - 1$.

$$(f \circ g)(x) = (x-1)^2 \quad (g \circ f)(x) = x^2 - 1.$$

Exercise 1.9 With the person next to you: Let $f(x) = x + 3$ and $g(x) = \frac{1}{x^2+1}$.

$$(f \circ g)(x) = \frac{1}{x^2+1} + 3 \quad (g \circ f)(x) = \frac{1}{(x+3)^2+1}.$$

3 Day 3: 10 Jan 2020

(1) Monday Quiz will be electronic or paper?

(2) Ask class if calculators will reduce stress. If so, then we'll allow calculators.

(3) Homework solutions will be posted on Friday instead of Monday - Suggestion of Kiran - I won't release at same time because people like me won't actually try and will instead look at the solutions right away; so there needs to be some time gap. If you're like me, then do them before I release the notes Friday (night)

1.1 Recall

- Even/Odd functions
- Increasing/decreasing functions
- $f + g$, $f - g$, fg, f/g
- Composition

1.2 Exponential functions - §1.4

An *exponential function* is a function of the form

$$f(x) = b^x$$

where b is a positive constant (*i.e.*, $b \in \mathbb{R}^+$).

How do we calculate b^x for every $x \in \mathbb{R}$?

x	b^x	Example
Integer ($\mathbb{Z}$)	b^x	if $x = 2$ then $b^x = b^2$ if $x = 0$ then $b^x = b^0 = 1$ if $x = -3$ then $b^x = b^{-3} = \frac{1}{b^3}$
Rational ($\mathbb{Q}$)	$b^{p/q} = \sqrt[q]{b^p}$	$x = \frac{3}{4}$ then $b^{3/4} = \sqrt[4]{b^3}$
Irrational ($\mathbb{R} \backslash \mathbb{Q}$)	Estimate using graph	$b^{\sqrt{2}} \approx b^{1.414}$

Example 1.10 Let $f(x) = 2^x$ and let $g(x) = x^2$.

$$(f \circ g)(x) = 2^{(x^2)} \quad (g \circ f)(x) = (2^x)^2$$

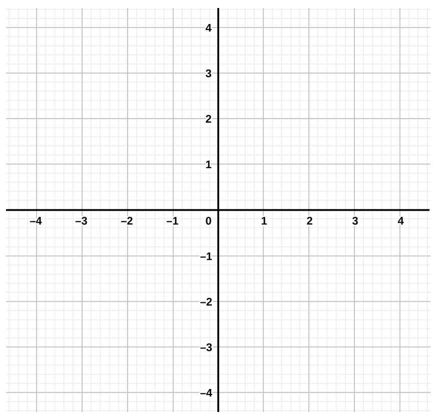
Are these the same function? (*i.e.*, is $f \circ g = g \circ f$) no try with $x = 1$.

raise hands to vote

1.3 Trigonometric functions - Appendix D

If any part of this section does not make sense, please raise your hand and force me to stop. Every country teaches trigonometry differently and at different times, so I literally have no idea what everyone knows/doesn't know.

Aram's favorite way to view trig functions:



draw circle - alphabetic $(\cos, \sin)$

This helps us calculate all the other trig functions as well:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \ \csc(\theta) = \frac{1}{\sin(\theta)}, \ \sec(\theta) = \frac{1}{\cos(\theta)}, \ \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Although we had the radius be equal to 1, we can alter the radius! Then we need a new method to solve for our trig functions. This is where an acronym comes in: sohcahtoa . This then gives us the following ways to calculate all the trig functions.

$$\sin(\theta) = \frac{opp}{hyp}, \ \cos(\theta) = \frac{adj}{hyp}, \ \tan(\theta) = \frac{opp}{adj}$$

There are a few trig functions that are worthwhile to know how to quickly calculate.

$$\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}$$

Using the triangles, we can also come up with some trig identities.

(1) $\sin^2(x) + \cos^2(x) = 1$ $\quad$ $opp^2 + adj^2 = hyp^2$ then divide by hyp^2

(2) $\tan^2(x) + 1 = \sec^2(x)$

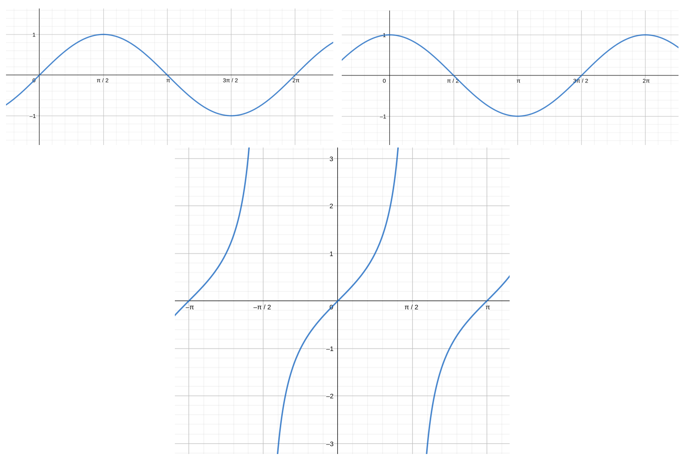
(3) $1 + \cot^2(x) = \csc^2(x)$

There are also addition formulas which I'll state, but won't go over to much.

(1) $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

(2) $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$ chuga chuga shoe shoe - choo choo train

We end this part by giving the graphs of all three functions:



1.4 Inverse functions - §1.5

A function allows us to map some number a to some number b using a formula, but what if we want to go backwards? How do we find a function that allows us to go from b to a, and is this even possible?

Example 1.11 Let's try an easy example. Let f be the function $f(x) = x + 1$. To go backwards we would need to use the formula $x - 1$.

But this doesn't always work!

Example 1.12 Let f be the function $f(x) = x^2$. If I know $f(x) = 4$, what is x? $\quad$ x can be either 2 or -2.

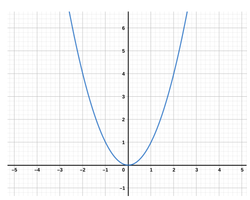
To be able to define an inverse, it seems as though we need a rule on our functions. A function f is *one-to-one* if

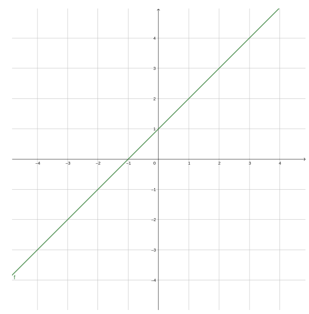
$$f(x) = f(y) \text{ implies } x = y$$

or alternatively

$$x \neq y \text{ implies } f(x) \neq f(y)$$

for every x and y in the domain.
If we have a graph there is a nice way to test this:





Proposition 1.13 *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

Definition 1.14 Let $f : A \to B$ be a one-to-one function with range B. Then its inverse function is the function $f^{-1} : B \to A$ with range A and is defined by

$$f^{(-1)}(y) = x \iff f(x) = y$$

for any $y \in B$.

This means that a function being one-to-one is dependent on its domain.

Example 1.15 For example, we saw $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = x^2$ is not one-to-one. But if we change our function's domain so that $f : \mathbb{R}^+ \to \mathbb{R}^+$ then $f(x) = x^2$ is one-to-one. Its inverse function is $\sqrt{x}$.

Week 2

13–17 Jan 2020

1 Day 1: 13 Jan 2020

2.1 Recall:

(1) Exponential functions

(2) Trig functions

(3) Inverse function

How do we find the inverse of a function?

(1) Let $y = f(x)$.

(2) Swap x and y in the formula.

(3) Solve for y.

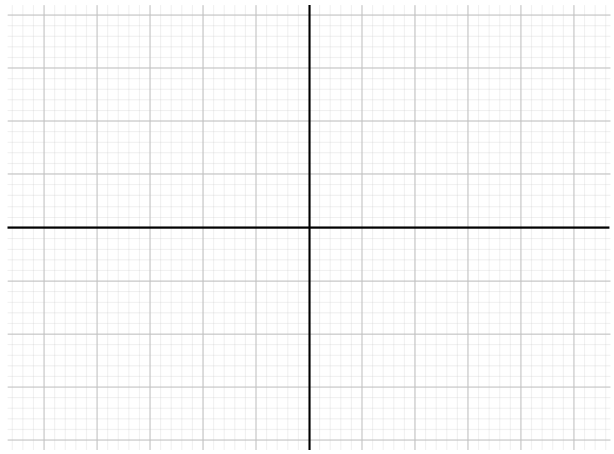
Example 2.1 If $f(x) = x^2$ then

(1) $y = f(x) = x^2$.

(2) $x = y^2$.

(3) $\sqrt{x} = y$.

Graphically, we can think of this as reflecting over the $y = x$ line.



give example

Inverse and reciprocal are not always the same!

$$f^{-1}(x) \neq (f(x))^{-1}$$

Exercise 2.2 With the person next to you: Let $f(x) = x^3 + 1$. What is its inverse function?

$$f^{-1}(x) = \sqrt[3]{x-1}$$

What is its reciprocal function?

$$(f(x))^{-1} = \frac{1}{x^3+1}$$

Another way to view this is by the *cancellation equations*:

$$f^{-1}(f(x)) = (f^{-1} \circ f)(x) = x \text{ and } f(f^{-1}(x)) = (f \circ f^{-1})(x) = x$$

There are two functions where finding the inverse becomes difficult: exponential and trigonometric functions. What are their inverses? For exponential, we let its inverse be called the *logarithm*. For $f(x) = b^x$ (where $b > 0$ and $b \neq 1$), the exponential passes the horizontal line test. The *logarithmic function with base b*, denoted $\log_b$, is the inverse of $f(x)$, *i.e.*, $f(x) = b^x$ implies $f^{-1}(x) = \log_b(x)$.

Example 2.3 Let $f(x) = 10^x$. Then $log_{10}(10^x) = x$ and $10^{\log_{10}(x)} = x$.

Some nice properties of logarithms:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$

- $\log_b(x^y) = y\log_b(x)$

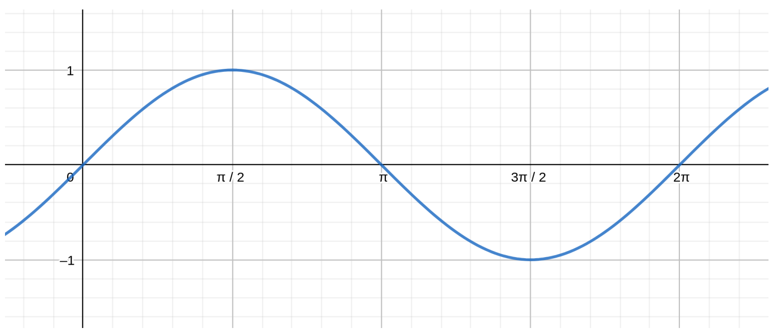
When $b = e$, the logarithm is called the *natural logarithm* and is denoted $\ln(x)$.

Some books will write $\log(x)$ (no subscript) to mean either $\log_{10}(x)$ or $\log_e(x) = \ln(x)$. Always double check with the book to ensure you know which version it is.

Trig functions are significantly more difficult. What is the inverse of the sine function?

Recall that you can just reflect over the $y = x$ line to find the inverse. So let's do that.

Exercise 2.4 Reflect $f(x) = \sin(x)$ over the $y = x$ line.



Turn to your neighbor and compare your drawing. Does this reflection give us an inverse function? no . How do we get an inverse function? restrict the range

The inverse function for sine is known as the *arcsine* function and is the function:

$$\sin^{-1}(x) = \arcsin(x) : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$$

Example 2.5 What is $\arcsin(\frac{\sqrt{3}}{2})$? 60° SohCahToa

We can similarly do the same thing for cosine! The inverse function for cosine is known as the *arccosine* function.

$$\cos^{-1}(x) = \arccos(x) : [-1, 1] \to [0, \pi]$$

$$\cos(x) : [0, \pi] \to [-1, 1]$$

Example 2.6 What is $\arccos(\frac{1}{2})$? 60°

Exercise 2.7 In groups of 3–5, solve the following:

(1) $\arcsin(\frac{1}{2}) = \underline{30°}$

(2) $\arccos(\frac{\sqrt{3}}{2}) = \underline{30°}$

(3) $\arcsin(1) = \underline{\frac{\pi}{2}}$

(4) $\arccos(1) = \underline{0}$

The inverse function for tangent is known as the *arctangent* function.

$$\tan^{-1}(x) = \arctan(x) : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$\tan(x) : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

There are 3 other trigonometric functions:

$$\csc = \frac{1}{\sin} \qquad \sec = \frac{1}{\cos} \qquad \operatorname{cotan} = \frac{1}{\tan}$$

Exercise 2.8 In the same group, solve the following:

(1) $\arctan(1) = \underline{45°}$.

(2) $\arctan(0) = \underline{0}$.

(3) $\operatorname{arccsc}(\sqrt{2}) = \underline{45°}$.

2 Day 2: 15 Jan 2020

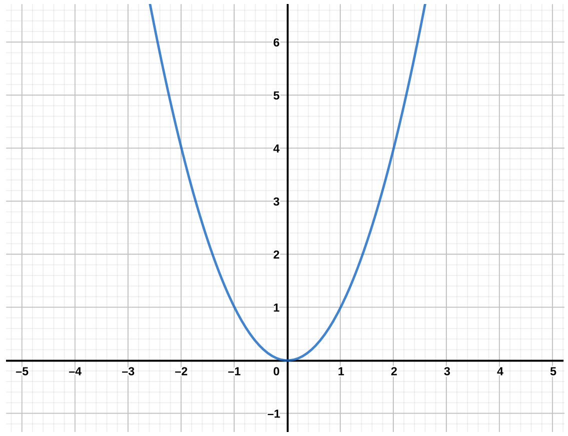
2.1 Recap:

(1) Inverse - exponential

(2) Inverse - trigs

2.2 A tangential problem - §2.1

Exercise 2.9 Suppose we have the function $f(x) = x^2$. How do we find the slope of the tangent line at $x = 1$? Do secants until you get a limit



With a partner, find the slope of the tangent line at

- $x = 0$
- $x = 2$
- $x = -1$

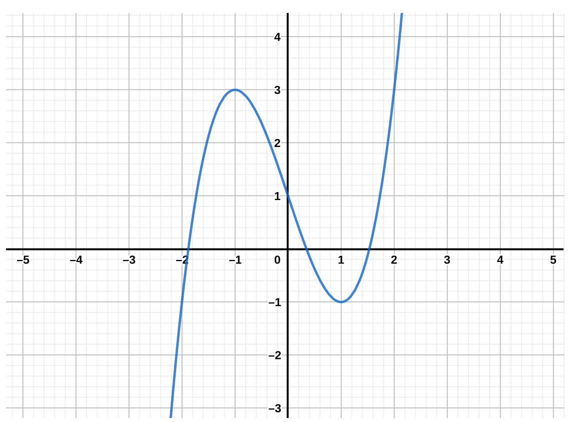
4hiddenSpace4cm Can you and your partner come up with a formula for the slope of the tangent line for arbitrary x?

$2x$

2.3 Limits - §2.2

What we just did can be formalized into what's called limits. We want to find the value of a function at a certain point without knowing the value at that point exactly.

Example 2.10 Let's first start with an example to help clarify. Let $f(x) = \underline{x^3 - 3x + 1}$.



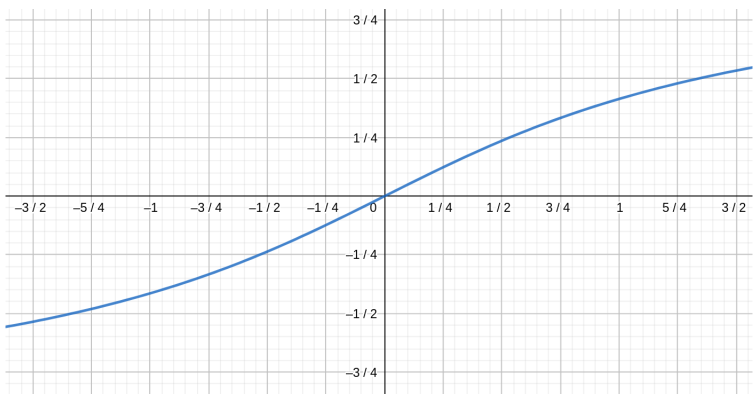
Try and calculate $f(\underline{\frac{3}{2}})$ without plugging in the number.

x	f(x)	x	f(x)
1	-1	2	3
1.4	-0.456	1.6	0.296
1.49	-0.162051	1.51	-0.087049
1.499	-0.1287455	1.501	-0.121245499
1.4999	-0.125374955	1.5001	-0.124624954999
1.49999	-0.12503749955	1.50001	-0.1249624994999

It looks like the value is approaching $\underline{-0.125 = -\frac{1}{8}}$. We write this as:

$$\lim_{x\to\frac{3}{2}} \left(x^3 - 3x + 1\right) = -\frac{1}{8}$$

Exercise 2.11 With a partner find the limit of the function $f(x) = \underline{\frac{\sqrt{x^2+1}-1}{x}}$ as x approaches $\underline{0}$. (Use a calculator!) Here's the graph:



x	f(x)	x	f(x)
-1	-0.414	1	0.414
-0.5	-0.236	0.5	0.236
-0.1	-0.0498756	0.1	0.0498756
-0.01	-0.004999875	0.01	0.004999875
-0.001	-0.0005	0.001	0.0005
-0.0001	-0.00005	0.0001	0.00005

What do you both think is the limit?

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x} = 0$$

Exercise 2.12 Again with your partner, try and find the limit of $f(x) = \frac{2x-2}{x^3-1}$ at $x = 1$.

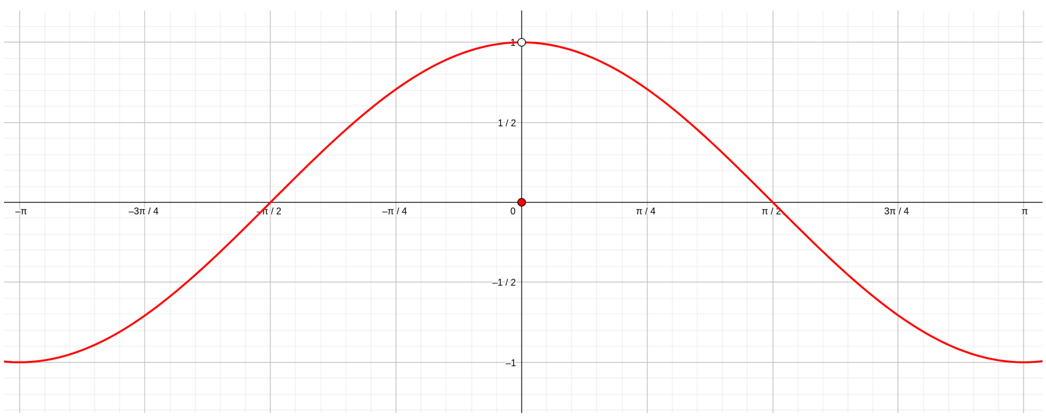
1.1	0.604...	0.9	0.738...
1.01	0.660044...	0.99	0.673378...
1.001	0.6660004...	0.999	0.667333778...
1.00001	0.6666600...	0.99999	0.66667333...

$$\lim_{x \to 1} \frac{2x-2}{x^3-1} = \frac{2}{3}$$

Example 2.13 Suppose we have the following piecewise function

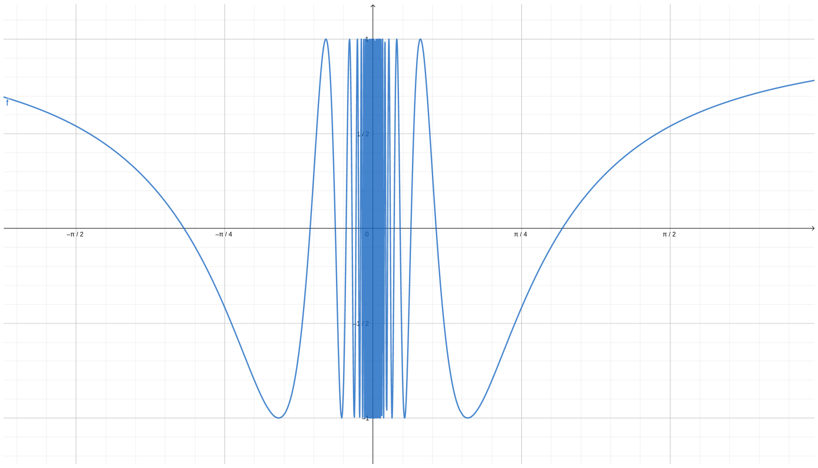
$$f(x) = \begin{cases} \cos(x) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and we want to know the limit at $x = 0$. Using the graph of the function, what do you think the limit is?



$$\lim_{x \to 0} f(x) = 1$$

Example 2.14 Let's look at something a little more complex. Say we have the function $f(x) = \cos(\frac{\pi}{2x})$ whose graph is the following:



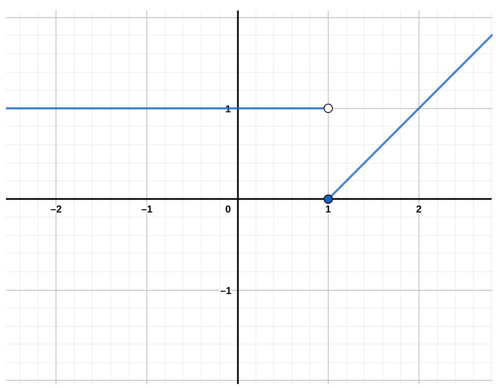
What is the limit at $x = 0$? Ask students to vote: 1, -1, 0, DNE

$$\lim_{x \to 0} f(x) = \underline{\text{does not exist}}$$

What happens if our function is split in two? Let

$$f(x) = \underline{\begin{cases} 1 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 1 \end{cases}}$$

whose graph is given by:



How can we calculate the limit at $x = 1$? What we can do is look at the limit from only one side instead of both sides like we had been doing. There are two sides of $x = 1$, a positive side and a negative side. Looking at the limit from the positive side, it looks like the limit is equal to $\underline{0}$. Looking at the limit from the negative side, it looks like the limit is equal to $\underline{1}$. We denote these by

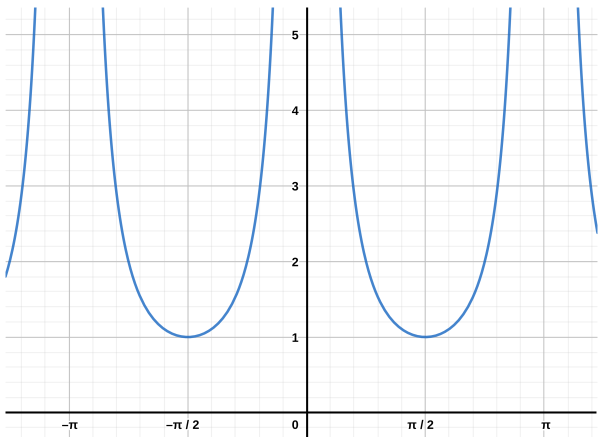
$$\lim_{x \to 1^+} f(x) = 0 \qquad \lim_{x \to 1^-} f(x) = 1$$

The "positive side" is exactly the numbers strictly greater than 1.

Comparing these limits with our original definition of limit, we see

$$\lim_{x\to a} f(x) = L \text{ if and only if } \underline{\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)}.$$

Let's look at another example. Suppose $f(x) = \underline{\frac{1}{\sin^2(x)}}$ with the following graph:



What is the limit as x approaches 0? The closer x gets to 0 (from either side) $\sin(x)$ gets closer to 0 and therefore $\frac{1}{\sin^2(x)}$ gets larger and larger. In fact, the limit does not approach any particular number, but keeps growing! To represent this, we say that the limit goes to infinity:

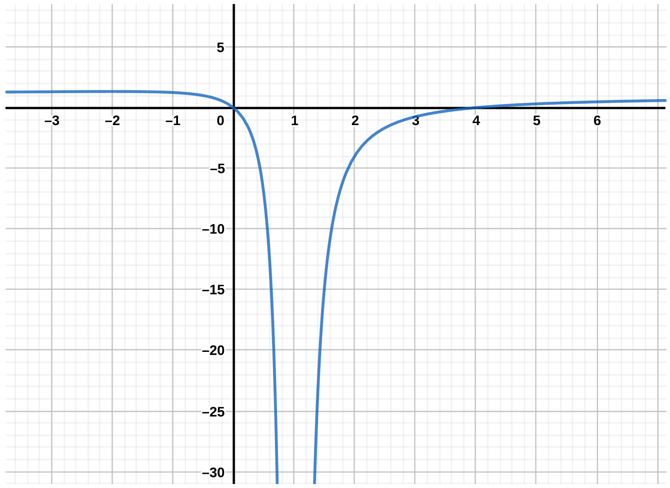
$$\lim_{x\to 0} \frac{1}{\sin^2(x)} = \infty$$

This does NOT mean that ∞ is a number! Nor are we saying that the limit exists. We are merely saying that as we approach $x = 0$, our function gets larger and larger and larger, without end.

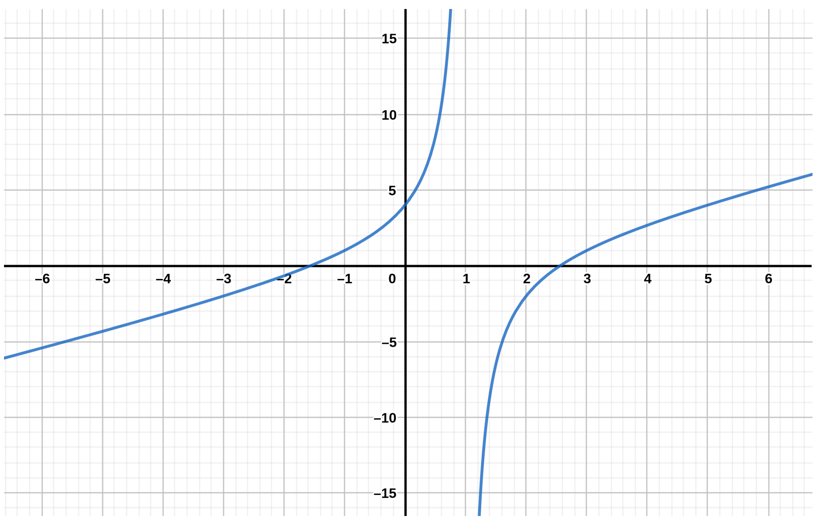
Similarly, if the function $f(x)$ gets smaller and smaller as we approach a point a, then we can say that the limit is negative infinity:

$$\lim_{x\to a} f(x) = -\infty$$

Example 2.15 Let $f(x) = \underline{\frac{x^2-4x}{(1-x)^2}}$ then as $x \to 1$ the limit goes to negative infinity.



Exercise 2.16 With a partner, let $f(x) = \underline{\frac{x^2-x-4}{x-1}}$ whose graph is given by:



What are the following limits:

$$\lim_{x\to 1^-} f(x) = \underline{\infty} \qquad \lim_{x\to 1^+} f(x) = \underline{-\infty} \qquad \lim_{x\to 1} = \underline{DNE}$$

3 Day 3: 17 Jan 2020

2.1 Recap:

(1) Tangent/Slope at point

(2) Limits

2.2 Precise definition of limit - §2.4

Let's be a little more precise by what we mean by a limit.

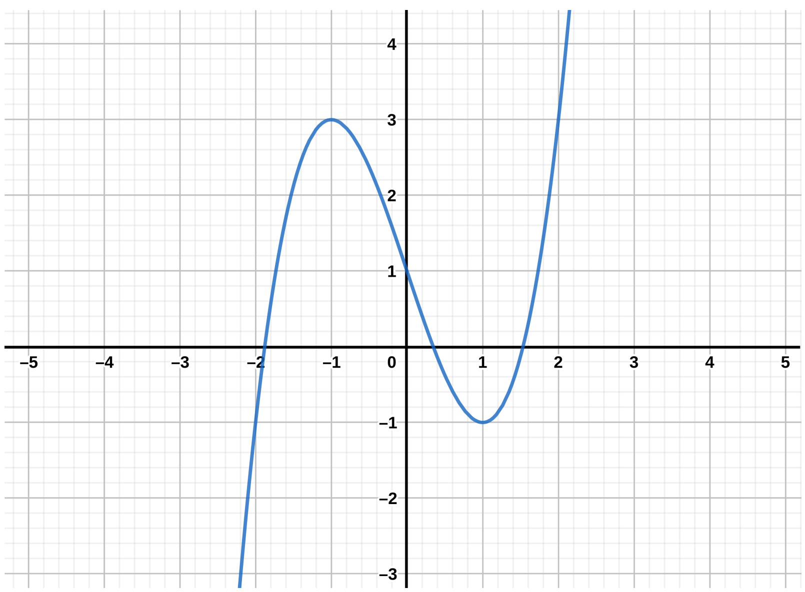
Definition 2.17 (Precise definition of a limit) Let f be a function defined on some open interval that contains a, but not necessarily defined at a. Then the *limit of* $f(x)$ *as* x *approaches* a *is* L, written

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \varepsilon$$

What does this mean?



We can use this definition as a way to show whether or not the intuitive solution to a limit problem is accurate or not. In order to prove that our intuition is correct, we use what's called an *epsilon-delta proof*. The following is a general outline of how an epsilon-delta proof works:

Epsilon-delta proof

(1) **Analysis:**

  (a) Go backwards! Start with $|f(x) - L| < \varepsilon$.

  (b) Try to change the formula to be of the form $0 < |x - a| < \delta$.

(2) **Proof:**

(a) Go forwards! Suppose $\varepsilon > 0$.
(b) Plug in the value of δ that you found in the analysis part.
(c) Solve until you find $|f(x) - L| < \varepsilon$.

(3) Finish up the proof.

Example 2.18 (Epsilon-delta proof) We want to prove that $\lim_{x\to 2}(3x+2) = 8$.

(1) **Analysis:**

(a) We start with $|f(x) - L| < \varepsilon$. In our case, $L = 8$ and $f(x) = 3x + 2$. Therefore

$$|(3x + 2) - 8| < \varepsilon$$

(b) We want to end with something like $0 < |x - 2| < ?$.

$$\begin{aligned} |(3x + 2) - 8| &= |3x - 6| \\ &= |3(x - 2)| \\ &= 3|x - 2| \\ &< \varepsilon \end{aligned}$$

In other words

$$|x - 2| < \frac{\varepsilon}{3}$$

So we let $\delta = \frac{\varepsilon}{3}$.

(2) **Proof:**

(a) Suppose $\varepsilon > 0$.
(b) Let $\delta = \frac{\varepsilon}{3} > 0$.
(c) If $0 < |x - 2| < \delta$, then

$$|(3x + 2) - 8| = |3x - 6| = 3|x - 2| < 3\delta = 3\frac{\varepsilon}{3} = \varepsilon.$$

(3) Therefore, by the definition of a limit

$$\lim_{x\to 2}(3x + 2) = 8.$$

Exercise 2.19 With a partner, prove that $\lim_{x\to 1}(5x - 2) = 3$.

Example 2.20 Let's do a significantly harder example. Prove that $\lim_{x\to 2} x^2 + x - 2 = 4$.

(1) **Analysis:**

(a) We first start with $|(x^2 + x - 2) - 4| < \varepsilon$.

(b) We now try and change the formula to $0 < |x - 2| < \delta$.

$$|x^2 + x - 2 - 4| = |x^2 + x - 6| = |(x + 3)(x - 2)| = |x + 3||x - 2|$$

Therefore,

$$|x - 2| < \frac{\varepsilon}{|x + 3|}$$

We can't set $\delta = \frac{\varepsilon}{|x+3|}$ since δ can't be in terms of x! We need to figure something out.

Let's set δ to 1 and see what happens. If $\delta = 1$ then $|x - 2| < 1$ implies $1 < x < 3$. And therefore $4 < x + 3 < 6$. But we're looking at $\frac{\varepsilon}{|x+3|}$ and therefore, since 6 is larger, this will make our fraction smaller. Therefore we let $\frac{\varepsilon}{|x+3|} < \frac{\varepsilon}{6}$. In other words we have the following two equations:

$$|x - 2| < 1 \text{ and } |x - 2| < \frac{\varepsilon}{|x + 3|} < \frac{\varepsilon}{6}$$

Therefore we can use $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}/$

(2) **Proof:**

(a) We let $\varepsilon > 0$.

(b) Let $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$.

(c) Suppose first that $\delta = 1$. Since δ is the min this implies $1 \leq \frac{\varepsilon}{6}$, in other words, $\varepsilon \geq 6$. Then

$$\begin{aligned}
0 < |x - 2| < \delta &\Rightarrow |x - 2| < 1 \\
&\Rightarrow |x - 2||x + 3| < |x + 3| \\
&\Rightarrow |x^2 + x - 6| < |x + 3| \\
&\Rightarrow |(x^2 + x - 2) - 4| < |x + 3| \\
&\Rightarrow |(x^2 + x - 2) - 4| < |x + 3| < 6 \leq \varepsilon
\end{aligned}$$

This completes the case $\delta = 1$.

Now suppose $\delta = \frac{\varepsilon}{6}$. We follow a similar approach.

$$\begin{aligned}
0 < |x-2| < \delta &\Rightarrow |x-2| < \frac{\varepsilon}{6} \\
&\Rightarrow |x-2||x+3| < \frac{\varepsilon}{6}|x+3| \\
&\Rightarrow |x^2+x-6| < \frac{\varepsilon}{6}|x+3| \\
&\Rightarrow |(x^2+x-2)-4| < \frac{\varepsilon}{6}|x+3| \\
&\Rightarrow |(x^2+x-2)-4| < \frac{\varepsilon}{6}|x+3| < \frac{\varepsilon}{6}6 = \varepsilon
\end{aligned}$$

This completes our second case and therefore the proof.

(3) Therefore $\lim_{x\to 2} x^2+x-2 = 4$ as desired.

Week 3

20–24 Jan 2020

1 Day 1: 20 Jan 2020

3.1 Recap:

(1) Precise defn of limit

(2) Epsilon-delta proof

Definition 3.1 (Left-hand and Right-hand limits) We say

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } a - \delta < x < a \text{ then } |f(x) - L| < \varepsilon$$

We say

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } a < x < a + \delta \text{ then } |f(x) - L| < \varepsilon$$

We use the exact same format to prove things in this case.

Example 3.2 Prove that $\underline{\lim_{x \to 8^+} \sqrt[3]{x} = 2}$.

(1) **Analysis:**

(a) Suppose

$$|\sqrt[3]{x} - 2| < \varepsilon$$

(b) We're eventually going to set $8 < x$ so we know $\sqrt[3]{x} > 2$. Therefore

$$|\sqrt[3]{x} - 2| = \sqrt[3]{x} - 2 < \varepsilon$$

In particular,

$$x < (\varepsilon + 2)^3 = \varepsilon^3 + 2\varepsilon^2 + 4\varepsilon + 8$$

Since we want

$$8 < x < 8 + \delta$$

Let $\delta = \varepsilon^3 + 2\varepsilon^2 + 4\varepsilon$.

(2) **Proof:**

(a) Suppose $\varepsilon > 0$.

(b) Let $\delta = \varepsilon^3 + 2\varepsilon^2 + 4\varepsilon > 0$.

(c) If $8 < x < 8 + \delta$, then

$$|\sqrt[3]{x} - 2| = \sqrt[3]{x} - 2 < \sqrt[3]{\delta + 8} - 2 = \sqrt[3]{\varepsilon^3 + 2\varepsilon^2 + 4\varepsilon + 8} - 2 = \sqrt[3]{(\varepsilon + 2)^3} - 2 = \varepsilon$$

(3) Therefore, by definition of the right-hand limit,

$$\lim_{x \to 8^+} \sqrt[3]{x} = 2.$$

One last precise limit definition!

Definition 3.3 (Infinite limits) Let f be a function defined on some open interval that contains the number a, except potentially not defined at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M, there is a positive number δ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } f(x) > M.$$

Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N, there is a positive number δ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } f(x) < N.$$

Example 3.4 Prove $\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty$.

(1) **Analysis:**

(a) Suppose
$$0 < M < \frac{1}{(x-2)^2}$$

(b) Then
$$\begin{aligned} M &< \frac{1}{(x-2)^2} \\ (x-2)^2 &< \frac{1}{M} \\ |x-2| &< \frac{1}{\sqrt{M}} \end{aligned}$$

Let $\delta = \frac{1}{\sqrt{M}}$

(2) **Proof:**

(a) Suppose $M > 0$.

(b) Let $\delta = \frac{1}{\sqrt{M}} > 0$.

(c) Then
$$|x-2| < \delta = \frac{1}{\sqrt{M}}$$
implies
$$M < \frac{1}{(x-2)^2} = f(x)$$

(3) Therefore, by definition of the infinite limit:
$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty$$

3.2 Limit laws - §2.3

In a lot of cases, the best way to know the limit is to plug in the number directly.

Theorem 3.5 *If f is a polynomial or a rational function and a is in the domain of f, then*
$$\lim_{x \to a} f(x) = f(a)$$

If we know the limit to a few functions at a point, then we can use that information to figure out the limit of their combinations!

Theorem 3.6 *Suppose c is some constant and suppose the following limits exist:*
$$\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)$$

Then

(1)
$$\lim_{x\to a}(f(x)+g(x)) = \underline{\lim_{x\to a} f(x) + \lim_{x\to a} g(x)}$$

(2)
$$\lim_{x\to a}(f(x)-g(x)) = \underline{\lim_{x\to a} f(x) - \lim_{x\to a} g(x)}$$

(3)
$$\lim_{x\to a}(cf(x)) = \underline{c\lim_{x\to a} f(x)}$$

(4)
$$\lim_{x\to a}(f(x)g(x)) = \underline{\left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)}$$

(5)
$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \underline{\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \textit{if } \lim_{x\to a} g(x) \neq 0}$$

(6)
$$\lim_{x\to a}(f(x))^n = \underline{\left(\lim_{x\to a} f(x)\right)^n} \textit{ when } n \in \mathbb{Z}^+$$

(7)
$$\lim_{x\to a}\sqrt[n]{f(x)} = \underline{\sqrt[n]{\lim_{x\to a} f(x)}}$$

class vote: This theorem requires the limits to exist. If limit is infinity, does it exist?

Another important rule helps us when one function is extremely similar to another function.

Theorem 3.7 *If $f(x) = g(x)$ when $x \neq a$ then, if $\lim_{x\to a} f(x)$ exists, then*

$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$$

We saw an example of this in Example 2.13.

Two more useful properties of limits are given next.

Theorem 3.8 *If $f(x) \le g(x)$ when x is near a (except possibly at a), and the limits of f and g both exist as x approaches a, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Theorem 3.9 (The squeeze theorem) *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{xtoa} g(x) = L$$

2 Day 2: 22 Jan 2020

3.1 Recap:

(1) Left/right hand limits

(2) Limit laws

The squeeze theorem is a super difficult theorem to use, but it is sometimes the easiest method of finding a limit!

Example 3.10 Let

$$f(x) = \underline{x^3 \cos\left(\frac{\pi}{2x}\right)}.$$

Show that $\lim_{x\to 0} f(x) = 0$. It would be super nice if we could just use one of our limit rules, but we found out in Example 2.14 that $\lim_{x\to 0} \cos\left(\frac{\pi}{2x}\right)$ does not exist. So, instead we're going to have to use the squeeze theorem.

One thing to notice is that cos always ossilates between -1 and 1. In other words:

$$-1 \leq \cos\left(\frac{\pi}{2x}\right) \leq 1$$

Multiply all sides by x^3 and we have

$$-x^3 \leq x^3 \cos\left(\frac{\pi}{2x}\right) \leq x^3.$$

Now we have something that looks like the squeeze theorem! Since both x^3 and $-x^3$ are polynomials and 0 is in their domains we have:

$$\lim_{x\to 0} -x^3 = \lim_{x\to 0} x^3 = 0^3 = 0$$

And, by the squeeze theorem (!) we know that

$$\lim_{x\to 0} f(x) = 0$$

Exercise 3.11 With a partner, use the squeeze theorem to show that

$$\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})} = 0$$

—

$$\begin{aligned} -1 \le \sin(1/x) \le 1 \\ e^{-1} \le e^{\sin(1/x)} \le e^1 \\ x^2/e \le x^2 e^{\sin(1/x)} \le ex^2 \\ \lim_{x\to 0} x^2/e = \lim_{x\to 0} ex^2 = 0 \\ \Rightarrow \quad \lim_{x\to 0} x^2 e^{\sin(1/x)} = 0 \end{aligned}$$

3.2 Continuity - §2.5

Limits are super easy when we can just plug in a number to get the answer. When we can just plug in a number a into a function $f(x)$ in order to get the limit we say that the function is continuous at a.

Definition 3.12 A function f is *continuous at a* if

$$\lim_{x\to a} f(x) = f(a)$$

For this equation to hold there are three things that normally need to be proved:

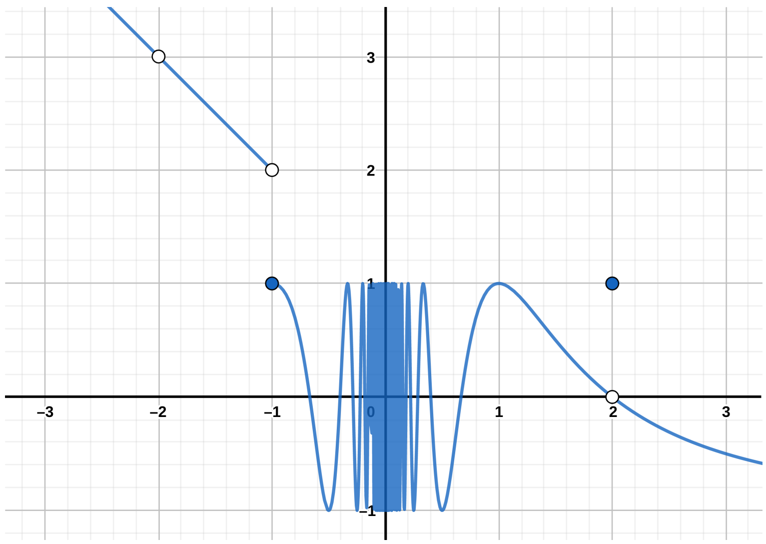
(1) $f(a)$ is defined!

(2) $\lim_{x\to a} f(x)$ exists. This means it can't be infinity!

(3) $\lim_{x\to a} f(x) = f(a)$.

If f is not continuous at a we say that it is *discontinuous at a*.

Example 3.13



Go over the different continuity types.

(1) Removable discontinuity: 1 fails

(2) Jump discontinuity: 2 fails

(3) Essential discontinuity: 2 fais

(4) Removable discontinuity: 3 fails

We can similarly define left-hand and right-hand continuity.

Definition 3.14 A function f is *continuous from the left at a* if

$$\lim_{x \to a^-} f(x) = f(a)$$

A function f is *continuous from the right at a* if

$$\lim_{x \to a^+} f(x) = f(a)$$

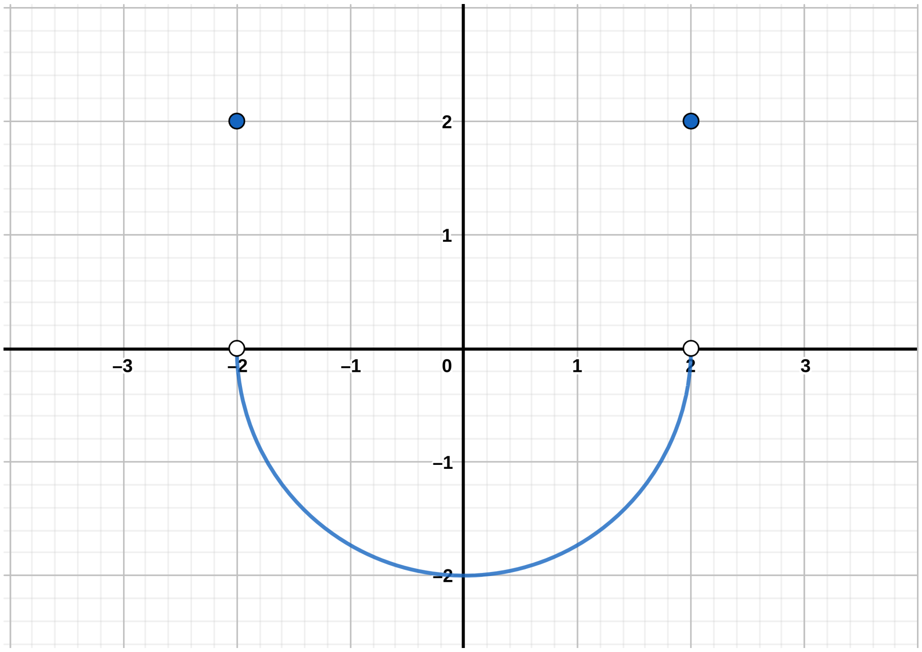
Similarly, we can do it on an interval!

Definition 3.15 A function f is *continuous on an interval I* if it is continuous at every number in the interval.

Example 3.16 Suppose

$$f(x) = \begin{cases} -\sqrt{4 - x^2} & \text{if } -2 < x < 2 \\ 2 & \text{if } x = -2 \text{ or } x = 2 \end{cases}$$

Show that f is continuous on the interval $(-2, 2)$, but discontinuous everywhere else. Here is a graph of the function:



First, to see that $f(x)$ is discontinuous outside of the closed interval $[-2, 2]$ notice that $f(a)$ is not defined outside of that interval! So automatically, it can't be continuous.

To see why $f(x)$ is discontinuous at 2 (similarly at -2) we need to show one of the three properties are false.

(1) $f(2) = 2$ - ✓

(2) $\lim_{x\to 2} f(x)$ exists? - No! It has a left-hand limit, but no right-hand limit.

To see that $f(x)$ is continuous at every point in $(-2, 2)$ we again look at our three properties. Let a be a number in $(-2, 2)$.

(1) $f(a) = -\sqrt{4 - a^2}$ which is well-defined since $a^2 \leq 4$.

(2) $\lim_{x\to a} f(x) =?$

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} -\sqrt{4 - x^2} \\
&= -\sqrt{\lim_{x\to a} 4 - x^2} \\
&= -\sqrt{4 - \lim x \to ax^2} \\
&= -\sqrt{4 - a^2} \\
&= f(a)
\end{aligned}$$

This gives us the whole proof.

Just like the limit laws, we have rules for continuity.

Theorem 3.17 *If f and g are continuous functions at a and c is some constant, then the following functions are also continuous at a:*

(1) $f+g$

(2) $f-g$

(3) cf

(4) fg

(5) $\frac{f}{g}$ *if* $g(a) \neq 0$

3 Day 3: 24 Jan 2020

3.1 Recap:

(1) Continuity

There are a lot of functions which are continuous!

(1) All polynomial functions are continuous everywhere.

(2) Rational functions are continuous on their domain.

(3) Trig functions are continuous on their domain.

(4) Inverse trig functions are continous on their domain.

(5) Exponential functions are continuous on their domain.

(6) Logarithmic functions are continuous on their domain.

Example 3.18 Where is the function $f(x) = \frac{x^3+2x^2-1}{71x-3}$ continuous? Since it's rational, it's continuous everywhere on its domain. It is defined everywhere except when $x = \frac{3}{71}$.

Exercise 3.19 With a partner, state where $f(x) = \frac{\ln(x)}{x^2-4}$ is continuous? $x > 0$ - $\ln(x)$ and $x \neq \pm 2$ for $x^2 - 4$

What about composition $f \circ g$?

Theorem 3.20 *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then*

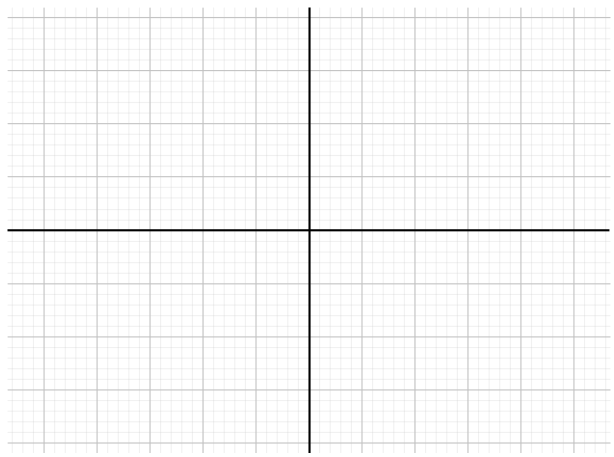
$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

Continuous functions have a nice property in that they go through every number.

Theorem 3.21 (Intermediate value theorem) *If f is continuous on the interval $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = N$.*

This is best visualized through a picture.



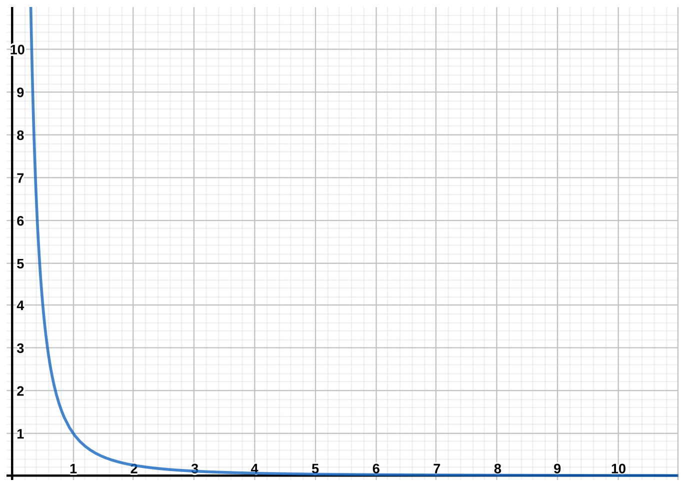
This theorem is super helpful for finding roots to polynomials!

Add online example of finding a root. See p123 for example

3.2 To infinity and beyond - §2.6

Although infinity is not a number, we can look at limits as a function gets really really big, *i.e.*, when the function goes toward infinity.

Example 3.22 Let $f(x) = \frac{1}{x^2}$ and notice that this function is always positive. Here is a graph of the function:



What this means is that the higher the value of x, the closer to 0 we are going to get. Intuitively we say that the limit of $f(x)$ as x goes to infinity is 0. We write this as:

$$\lim_{x\to\infty} f(x) = 0$$

We can play this game with any function. If f is a function defined on (a, ∞) then

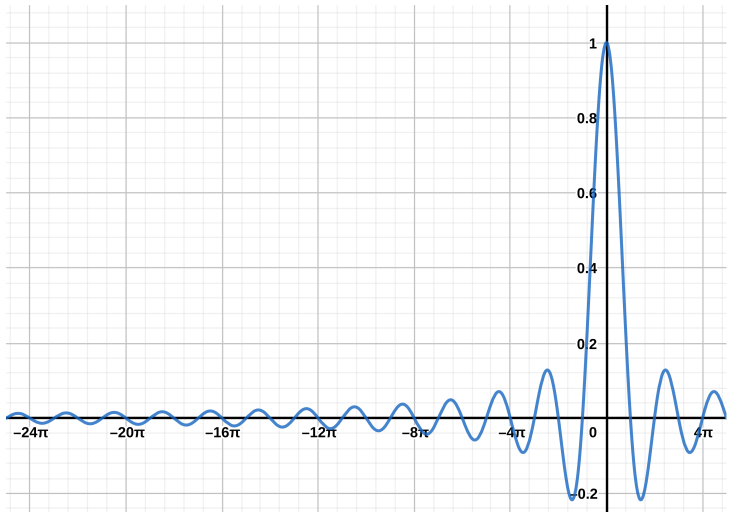
$$\lim_{x\to\infty} f(x) = L$$

means that the value of $f(x)$ approaches L as x gets larger and larger.

Similarly, you can do the same thing for negative infinity.

$$\lim_{x\to-\infty} f(x) = L$$

Example 3.23 Let $f(x) = \frac{\sin(x)}{x}$ with the graph:



What is

$$\lim_{x\to-\infty} f(x) = \underline{0}$$

The line $y = L$ is called a *horizontal asymptote* of the curve $f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

There is a battle in mathematics on the definition of a horizontal asymptote! The majority give the definition above, but some authors additionally require that the line $f(x)$ doesn't cross the asymptote! our $-\infty$ example would not be a horizontal asymptote!

Exercise 3.24 Let $f(x) = \underline{\frac{1}{x}}$. With a partner, find

$$\lim_{x\to\infty} f(x) = \underline{0} \text{ and } \lim_{x\to-\infty} f(x) = \underline{0}$$

Week 4

27–31 Jan 2020

1 Day 1: 27 Jan 2020

4.1 Recap:

(1) Continuous functions

(2) Limits to infinity

Based off last week, what do you think the following theorem should say?

Theorem 4.1 *If $r > 0$ is a rational number such that x^r is defined for all x, then*

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = \underline{0}$$

This theorem is ridiculously strong! Let's look at an example.

Example 4.2 Let $f(x) = \underline{\frac{4x^5+3x^2+x-77}{7x^5-6x^3+22x^2-943}}$. What is

$$\lim_{x\to\infty} f(x) = \underline{\frac{4}{7}}?$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x^5+3x^2+x-77}{7x^5-6x^3+22x^2-943} &= \lim_{x\to\infty} \frac{\frac{4x^5+3x^2+x-77}{x^5}}{\frac{7x^5-6x^3+22x^2-943}{x^5}} \\
&= \lim_{x\to\infty} \frac{4+\frac{3}{x^3}+\frac{1}{x^4}-\frac{77}{x^5}}{7-\frac{6}{x^2}+\frac{22}{x^3}-\frac{943}{x^5}} \\
&= \frac{\lim_{x\to\infty}\left(4+\frac{3}{x^3}+\frac{1}{x^4}-\frac{77}{x^5}\right)}{\lim_{x\to\infty}\left(7-\frac{6}{x^2}+\frac{22}{x^3}-\frac{943}{x^5}\right)} \\
&= \frac{\lim_{x\to\infty} 4+\lim_{x\to\infty}\frac{3}{x^3}+\lim_{x\to\infty}\frac{1}{x^4}+\lim_{x\to\infty}-\frac{77}{x^5}}{\lim_{x\to\infty} 7+\lim_{x\to\infty}-\frac{6}{x^2}+\lim_{x\to\infty}\frac{22}{x^3}+\lim_{x\to\infty}-\frac{943}{x^5}} \\
&= \frac{4+0+0+0}{7+0+0+0} \\
&= \frac{4}{7}
\end{aligned}$$

Example 4.3 Evaluate

$$\lim_{x\to\infty} 2x-\sqrt{4x^2+1}.$$

First, notice that we can't use our limit laws since $\lim_{x\to\infty} 2x = \infty$ which is not a number! Therefore the limit does not exist and we can't use our laws. So instead we need to use a trick.

$$\begin{aligned}
\lim_{x\to\infty} 2x-\sqrt{4x^2+1} &= \lim_{x\to\infty}\left(2x-\sqrt{4x^2+1}\right)\frac{2x+\sqrt{4x^2+1}}{2x+\sqrt{4x^2+1}} \\
&= \lim_{x\to\infty}\frac{4x^2-4x^2+1}{2x+\sqrt{4x^2+1}} \\
&= \lim_{x\to\infty}\frac{-1}{2x+\sqrt{4x^2+1}} \\
&= 0
\end{aligned}$$

Example 4.4 Evaluate

$$\lim_{x\to\infty}\cos(x) = DNE$$

Draw Cos

Example 4.5 Evaluate

$$\lim_{x\to\infty} x = \infty$$

Draw graph

Example 4.6 Evaluate

$$\lim_{x\to\infty} x^3 - x^2 = \lim_{x\to\infty} x^2(x-1) = \infty$$

Note that we can't do

$$\lim_{x\to\infty} x^3 - x^2 = \infty - \infty$$

Exercise 4.7 With a partner try and evaluate

$$\lim_{x\to\infty} \frac{x^2 - x}{2x - 5}$$

$$\begin{aligned} &= \lim_{x\to\infty} \frac{x-1}{2-\frac{5}{x}} \\ &= \lim_{x\to\infty} \infty \end{aligned}$$

Definition 4.8 (Precise definition of limit at infinity) Let f be a function defined on the interval (a, ∞) for some number a. Then

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if } x > N \text{ then } |f(x) - L| < \varepsilon$$

Let f be a function defined on the interval $(-\infty, a)$ for some number a. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if } x < N \text{ then } |f(x) - L| < \varepsilon$$

Let f be a function defined on the interval (a, ∞) for some number a. Then

$$\lim_{x\to\infty} f(x) = \infty$$

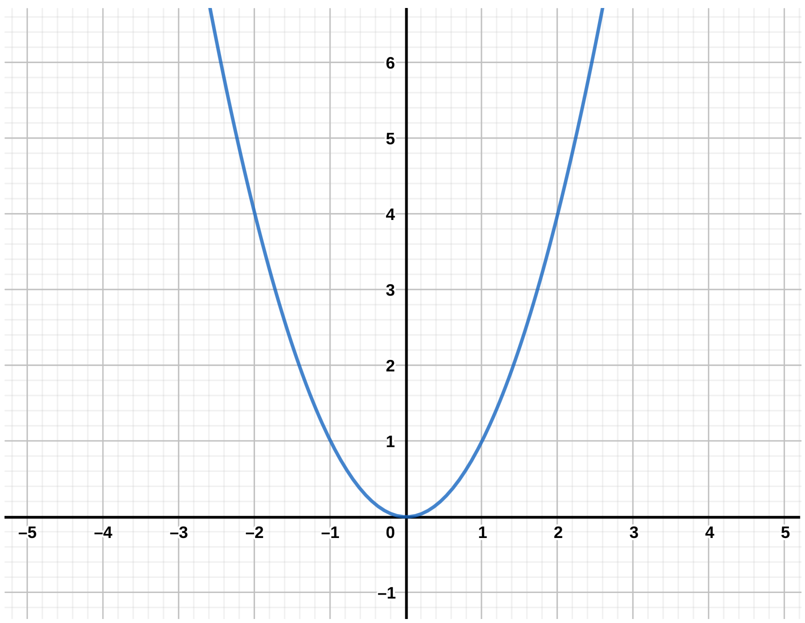
means that for every positive number M there is a corresponding positive number N such that

$$\text{if } x > N \text{ then } f(x) > M$$

4.2 Derivatives - §2.7

Remember how at the beginning of limits we looked at a tangent line to a curve. We can now use all the technology we've developed to answer this question for any arbitrary curve!

Let's look at that very first exercise. Let $f(x) = x^2$, what is the slope of the tangent line at $x = 1$?



Use graph! Recall that the method we used was to find secants to the curve. So we used equations like:

$$\frac{f(x) - f(a)}{x - a}$$

If we keep moving x closer and closer to a then we will eventually find the tangent line. In other words, the tangent line at point $(a, f(a))$ has as slope:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

and from there you can figure out the tangent line for any given point!

Exercise 4.9 Use this method to find the tangent line $y = mx + b$ when $a = \underline{3}$ with a partner. Explain what you mean by "this method" as it's ambiguous.

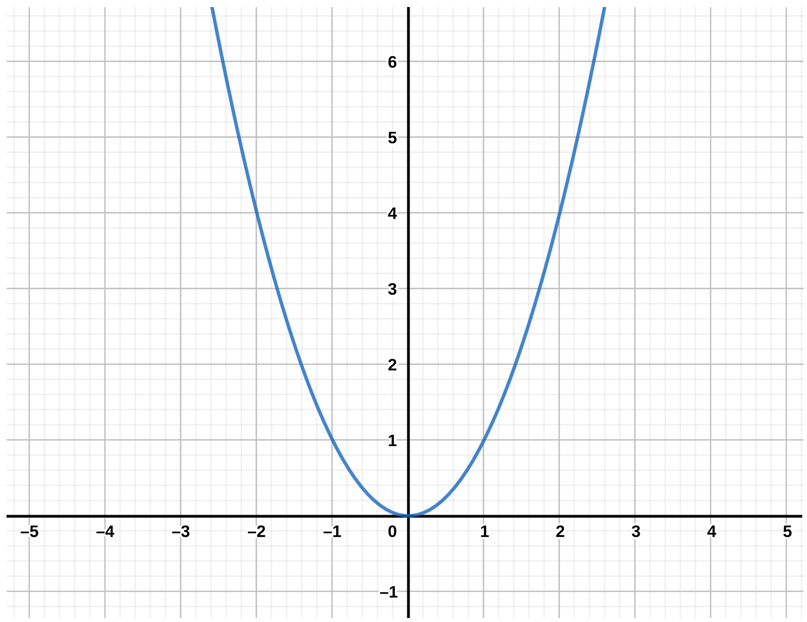
"plug in values"

$$\text{Point: } (3, 3^2) = (3, 9)$$

$$m = \lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 3} x + 3 = 6$$

$$y = mx + b \Rightarrow 9 = 3 * 6 + b \Rightarrow b = -9 \Rightarrow y = 6x - 9$$

A lot of books (like ours) find the slope in another method. They fix a distance away from the point a and they try and decrease this distance. What does this mean?



Start with some point a where you want to find the tangent line. You let h be some distance away from the point a. Then you try and decrease the distance and solve the following limit:

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

draw a,h, etc. on graph above

Sometimes, this method is easier to solve than the previous version.

Example 4.10 Let $f(x) = 1/x^2$ and find the slope of the tangent line at $(2, \frac{1}{4})$.

In this case, we do as before an just plug in

$$\begin{aligned}
m &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{4}}{h} \\
&= \lim_{h\to 0} \frac{\frac{4-(2+h)^2}{4(2+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{4-4-4h-h^2}{4h(2+h)^2} \\
&= \lim_{h\to 0} -\frac{4h+h^2}{4h(2+h)^2} \\
&= \lim_{h\to 0} -\frac{4+h}{4(2+h)^2} \\
&= -\frac{4+0}{4(2+0)^2} \\
&= -\frac{4}{16} \\
&= -\frac{1}{4}
\end{aligned}$$

This slope line has a special name and it is (kinda) the whole point of this course.

Definition 4.11 The *derivative of a function f at a number a* is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

If you want, you can use the other definition as well (the $x \to a$ one).

2 Day 2: 29 Jan 2020

4.1 Recap:

(1) Theorem $1/x^r$

(2) Precise defn - lim at ∞

(3) Derivatives

4.2 Rates of change - §2.7

We now are going to talk about some applications of the derivative. In particular, we're going to talk about rates of change.

We'll start off today on a rocket ship. You are tasked with helping figure out whether or not, at any point, the acceleration of the rocket is to much for the human body to handle (we don't want to kill off the astronauts!) A g-force is calculated at $9.8\frac{m}{s^2}$ and your scientists say that you don't want more than 3g for an astronaut. In other words, your acceleration can't go above $3 \cdot 9.8 = 29.4$.

You go and talk with the pilot and they say it will take roughly 120 minutes to reach escape velocity (the velocity needed for you to escape the earth's atmosphere) and that your velocity is given by the function $f(x) = \underline{x^{\frac{3}{2}}}$.

Will the astronauts survive?

What we need to do is calculate the acceleration of the astronauts/space shuttle. Acceleration is just the rate at which something changes velocity. So what we need to do is calculate the rate of change!

Definition 4.12 Let $y = f(x)$ be a function. The change in x from x_1 to x_2 is given by

$$\Delta x = x_2 - x_1.$$

The change in y from x_1 to x_2 is given by

$$\Delta y = f(x_2) - f(x_1)$$

The *average rate of change* is the quotient of these two changes:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This probably looks familiar! But we need to know whether acceleration at any given moment is going to exceed 29.8, not just the average rate of change! The limit will give us the acceleration.

Definition 4.13 The *instantaneous rate of change* of a function $y = f(x)$ is given by

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The instantaneous rate of change of velocity is acceleration.

So let's see if our astronauts will survive. We need to caclulate

$$\lim_{\Delta x \to 0} \frac{f(x_2) - f(x_1)}{\Delta x}$$

Let's make this look ike our previous equation.

$$h = \Delta x,\ a = x_1,\ \text{ and } a + h = x_1 + x_2 - x_1 = x_2$$

So we have

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let's solve

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{(a+h)^{\frac{3}{2}} - a^{\frac{3}{2}}}{h} \\
&= \lim_{h \to 0} \frac{(a+h)^{\frac{3}{2}} - a^{\frac{3}{2}}}{h} \frac{(a+h)^{\frac{3}{2}} + a^{\frac{3}{2}}}{(a+h)^{\frac{3}{2}} + a^{\frac{3}{2}}} \\
&= \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h\left((a+h)^{\frac{3}{2}} + a^{\frac{3}{2}}\right)} \\
&= \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h\left((a+h)^{\frac{3}{2}} + a^{\frac{3}{2}}\right)} \\
&= \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3}{h\left((a+h)^{\frac{3}{2}} + a^{\frac{3}{2}}\right)} \\
&= \lim_{h \to 0} \frac{3a^2 + 3ah + h^2}{(a+h)^{\frac{3}{2}} + a^{\frac{3}{2}}} \\
&= \frac{3a^2}{2a^{\frac{3}{2}}} \\
&= \frac{3a^{\frac{1}{2}}}{2}
\end{aligned}$$

In other words, the acceleration is given by the function

$$acc(x) = \frac{3}{2}\sqrt{x}$$

Since our time is max at $x = 120$, let's plug in 120 and see what we get. To make life easier, let's look at $x = 144 = 12^2$.

$$acc(144) = \frac{3}{2}\sqrt{144} = \frac{3}{2}12 = 3 * 6 = 18$$

In other words, the astronauts survive!

There are tons of examples of rate of change in real life. You can look at the book for more examples or you can do a google search. The instantaneous rate of change (aka the derivative) is the most crucial part of this and so we'll start exploring the derivative even further.

4.3 The derivative - §2.8

Recall that we've so far been looking at the derivative of a function f at a fixed number a
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$
What if we want to look at the derivative at an arbitrary point, not just at a? For this we replace the number a by a variable.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

We already did this when we were looking at acceleration. Notice that this allows us to look at f' as a function! This function, f', is called the *derivative of f*.

Example 4.14 Let's do a test run with a super easy example. Find the derivative of $f(x) = x$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)-x}{h} \\
&= \lim_{h\to 0} \frac{h}{h} \\
&= \lim_{h\to 0} 1 \\
&= 1
\end{aligned}$$

Therefore $f'(x) = 1$.

Exercise 4.15 Try a more complicated example with a partner. Find the derivative of $f(x) = x^2 - x$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2-(x+h)-x^2+x}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2-x-h-x^2+x}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2-h}{h} \\
&= \lim_{h\to 0} 2x+h-1 \\
&= 2x-1
\end{aligned}$$

There are a lot of different ways that people write the derivative $f'(x)$. This comes from the history of derivation! $f'(x)$ is by Lagrange (1770). $\dot{y}$ is by Newton (1642-1726). $\frac{dy}{dx}$ is by Leibniz (1646-1716). $Df = \frac{df}{dx}$ is by Euler (Actually from Arbogast - 1800). Massive controversy over who invented calculus! Starting in 1699, the math community was in an uproar. Leibniz published his calculus book in 1684, but Newton accused him of stealing his unpublished ideas. This argument went full out in 1711 when European mathematicians and English mathematicians basically stopped talking to one another and there was the great mathematical divide. We're now pretty sure they both came up with their ideas independently, it just happened to coincide.

$$f'(x) = \dot{y} = \frac{dy}{dx} = \frac{df}{dx} = Df$$

Fractions are not ratios!!! You can't just divide

Definition 4.16 We say that a function f is *differentiable at a* if $f'(a)$ exists.

Example 4.17 Where is the following function differentiable?

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

First let's look at $x > 0$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{1-1}{h} \\ &= 0 \end{aligned}$$

So it is differentiable at $x > 0$.

If $x < 0$, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-1+1}{h} \\ &= 0 \end{aligned}$$

So it is differentiable at $x < 0$.

What happens at $x = 0$? Let's look at the limit from both sides.

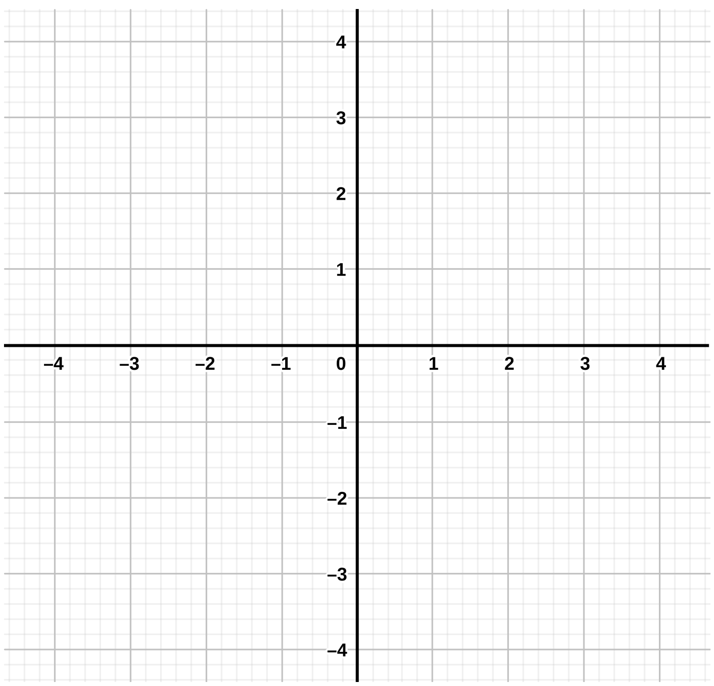
$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{1-1}{h} = 0$$
$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{-1-1}{h} = \infty$$

Therefore, the limit does not exist and our function is not differentiable at $x = 0$.

Being differntiable, actually tells us whether our function is continuous or not!

Theorem 4.18 *If f is differentiable at a, then f is continuous at a.*

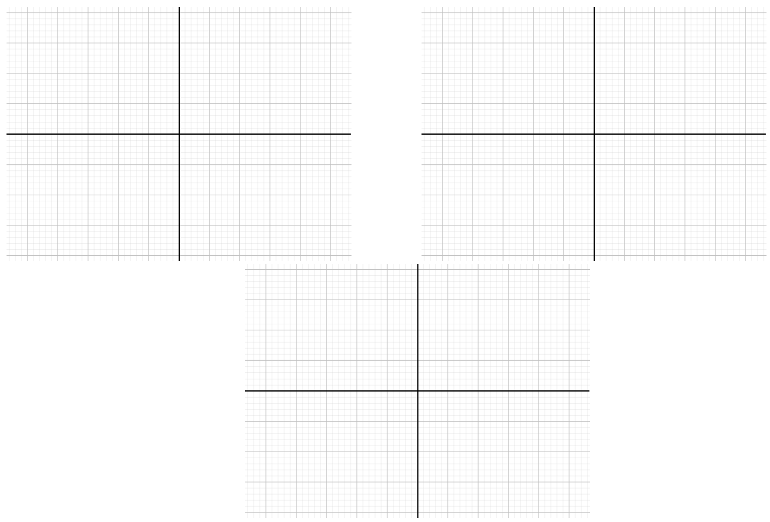
We see this already in our example!



draw function and show it's not cont at 0.

The opposite direction is *not* true! There are functions that are continuous at a point but not differentiable! There is an example of such a function in the book.

Intuitively, there are three main cases when a function is not differentiable:



Corner, discontinuity, vertical tangent

4.4 Higher derivatives

Recall that when we were in outerspace that acceleration was calculated as the instantaneous rate of change (aka derivative) of the velocity. Velocity itself is also a rate of change! It is the instantaneous rate of change of position. So if velocity is the derivative of position, and acceleration is derivative of velocity then acceleration is the derivative of the derivative of position! There are two derivatives!

This happens because we're able to view the derivative as a function itself. And since it's a function, we can ask what is the derivative of the derivative. This is normally called the *second derivative*. In a similar fashion we can define the *third derivative*, *fourth derivative*, *nth derivative*, etc. Notation-wise, this is what they look like:

$$\begin{aligned}
2^{nd}: &\quad f''(x) = \ddot{y} = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = D^2f \\
3^{rd}: &\quad f'''(x) = \dddot{y} = \frac{d^3y}{dx^3} = \frac{d^3f}{dx^3} = D^3f \\
4^{th}: &\quad f''''(x) = f^{iv}(x) = f^{(4)}(x) = \ddddot{y} = \overset{4}{\dot{f}} = \frac{d^4y}{dx^4} = \frac{d^4f}{dx^4} = D^4f \\
n^{th}: &\quad f^{(n)}(x) = \overset{n}{\dot{f}} = \frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = D^nf
\end{aligned}$$

Exercise 4.19 With a partner, find the third derivative of $f(x) = x^2$.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x$$
$$f''(x) = \lim_{h\to 0} \frac{2(x+h) - 2x}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$
$$f'''(x) = \lim_{h\to 0} \frac{2-2}{h} = 0$$

3 Day 3: 31 Jan 2020

4.1 Recap:

(1) Rates of Change

(2) Derivatives

(3) Differentiable

(4) Higher Derivatives

Practice Midterm 1

Week 5

3–7 Feb 2020

1 Day 1: 3 Feb 2020

Midterm 1

2 Day 2: 5 Feb 2020

5.1 Recall

5.2 Derivatives of polynomials - §3.1

All of the above is fun, but this can get really complicated, really quickly. So instead, let's try and generalize our results so that we don't have to work as hard.

From our previous examples we see that:

$$\frac{d}{dx}(x^2) = \underline{2x^1} \qquad \frac{d}{dx}(x) = \underline{1x^0}$$

We're tempted to state the following (which turns out to be true!)

Theorem 5.1 *If n is a real number, then*

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof. We're actually gonna prove this one for positive integers, because it's nice and (relatively) easy. Two parts:

(1) $\underline{x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}$. Prove the above to yourself.

(2)

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x\to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x\to a} \left(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}\right) \\ &= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

□

Example 5.2
- What is the derivative of $f(x) = x^{93}$? $\underline{93x^{92}}$.
- What is the derivative of $f(x) = \sqrt[5]{x^3}$? $\underline{\frac{3}{5}x^{\frac{-2}{5}}}$.
- What is the derivative of $f(x) = x^{\pi}$? $\underline{\pi x^{\pi - 1}}$.

Using our limit laws we have the following theorem:

Theorem 5.3 *Let f and g be differentiable functions and c a constant.*

- $\underline{\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)}$
- $\underline{\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)}$
- $\underline{\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)}$

This means we can differentiate more complicated polynomials!

Example 5.4 What is the derivative of $f(x) = x^9 + 3x^4 - x^2 + 1$? $9x^8 + 12x^3 - 2x$

Example 5.5 Let's apply this to something real world. Suppose we have a rod (or a piece of wire) of metal. The rod is "homogeneous" if the density of the rod is linear throughout. In other words it is defined to be the mass (kg) per unit length (m). We can write this as ρ (density) is equal to the mass m divided by length ℓ:

$$\rho = \frac{m}{\ell}$$

But, now suppose our rod is not homogeneous. Instead, suppose that the mass is given in some function $m = f(x)$. Then, the average density is:

$$\rho = \frac{\Delta m}{\Delta \ell} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This looks exactly like the derivative! So then the density at any given point is given by:

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta \ell} = \frac{dm}{dx}$$

For example, let $f(x) = x^{3/2}$. Using the limit rules we just saw, we know $f'(x) = \underline{\frac{3}{2}x^{1/2}}$. In other words at any point x on our rod, $\rho = \underline{\frac{3}{2}x^{1/2}}$.

So if we look 1 metre up our rod, then we know the density is:

$$\frac{3}{2}1^{1/2} = \frac{3}{2}\text{kg/m}$$

5.3 Exponential functions - §3.1

Let's talk about exponential functions and what their limits look like. Recall that an exponential function is a function of the form $f(x) = b^x$ for b a real number. Let's calculate its derivative.

$$\begin{aligned}
f'(x) &= \underline{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}} \\
&= \underline{\lim_{h\to 0} \frac{b^{x+h}-b^x}{h}} \\
&= \underline{\lim_{h\to 0} \frac{b^x b^h - b^x}{h}} \\
&= \underline{b^x \lim_{h\to 0} \frac{b^h - 1}{h}} \\
&= \underline{b^x \lim_{h\to 0} \frac{b^h b^0 - b^0}{h}} \\
&= b^x f'(0)
\end{aligned}$$

But now we need to figure out what $f'(0)$ is! There is no easy way to do this, so we need to go back to our old methods of looking at multiple values of h.

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$
0.1	0.7177346...	1.1612317...
0.01	0.695555...	1.104669...
0.001	0.693387...	1.099216...
0.0001	0.693171...	1.0098673...

It turns out that

$$\begin{aligned}
b = 2 \Rightarrow f'(0) &= \lim_{x\to 0} \frac{2^x - 1}{x} = \underline{\ln(2)} \approx 0.69314718\ldots \\
b = 3 \Rightarrow f'(0) &= \lim_{x\to 0} \frac{3^x - 1}{x} = \underline{\ln(3)} \approx 1.098612289\ldots
\end{aligned}$$

So, we have

$$\frac{d}{dx} 2^x = \underline{\ln(2) 2^x} \qquad \frac{d}{dx} 3^x = \underline{\ln(3) 3^x}$$

and in general:

$$\frac{d}{dx} b^x = \underline{\ln(b) b^x}$$

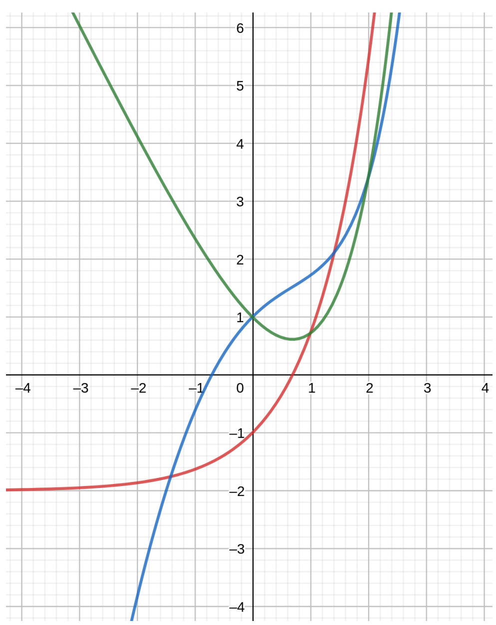
What would be really cool is if we could find a number where $f'(0) = 1$. This would imply $\underline{f'(x) = f'(0)b^x = b^x = f(x)}$!! This is where the number $\underline{e}$ comes from. If $\underline{b = e}$ then $\underline{f(x) = f'(x)}$ and we are happy.

Example 5.6 Find the second derivative of $f(x) = e^x - x^2$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(e^x - x^2\right) \\
&= \frac{d}{dx}e^x - \frac{d}{dx}x^2 \\
&= e^x - 2x \\
f''(x) &= \frac{d}{dx}\left(e^x - 2x\right) \\
&= \frac{d}{dx}e^x - 2\frac{d}{dx}x \\
&= e^x - 2
\end{aligned}$$

$$f''(x) = e^x - 2$$

The graph of these functions is the following:



Example 5.7 Let's look at an example from biology. Suppose we're looking at the population growth of some animal over time. If we say that the number of animals at any given time is given by the function $n = f(t)$ then we can

calculate the average rate of growth by the change in the number of animals over some given time:

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Notice how (again) this looks like the definition of the derivative!

If we want to then know what the rate of growth at any given time (aka the instantaneous rate of growth), then we just need to take the limit as the change in time goes to 0:

$$\text{instantaneous rate of growth} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Suppose our growth is given by $f(t) = 2e^t + 3t$. What is the instantaneous rate of growth?

We know from this section that $f'(t) = \underline{2e^t + 3}$. Therefore, if want to know how much the population is growing in exactly one year then:

$$f'(1) = 2e^1 + 3 = 2e + 3$$

3 Day 3: 7 Feb 2020

5.1 Recall:

5.2 Product and quotient rules - §3.2

Recall that when we did addition and subtraction of derivatives, we saw that a very similar thing happened as with the limit laws; we could just separate them. For the limit laws we had:

$$\lim_{x \to a} (f(x) + g(x)) = \left(\lim_{x \to a} f(x)\right) + \left(\lim_{x \to a} g(x)\right)$$

And for derivatives we had:

$$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

We now ask the same question for multiplication since we have the same limit law:

$$\lim_{x\to a}\left(f(x)g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$

It turns out the same property does *not* hold for multiplication (and division)! In particular we have $(fg)' \neq f'g'$ like we would want.

Example 5.8 Let's look at a super simple example for proof. Let $f(x) = x$ and $g(x) = x$. Then,

$$f'(x) = g'(x) = 1 \quad \text{and} \quad f(x)g(x) = x^2$$

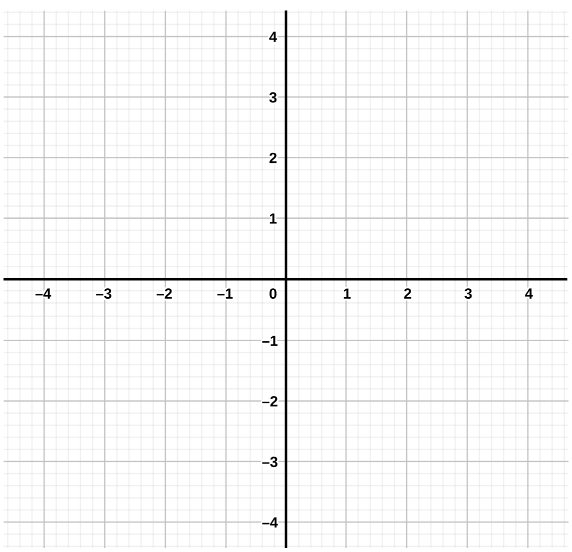
Then

$$(fg)' = \frac{d}{dx}x^2 = 2x \quad \text{and} \quad f'g' = 1 \cdot 1 = 1$$

and $1 \neq 2x$ as functions, therefore this doesn't work.

So we need to figure something else out to calculate $(fg)'$. This is a bit complicated, but it turns out the best way is to use geometry.

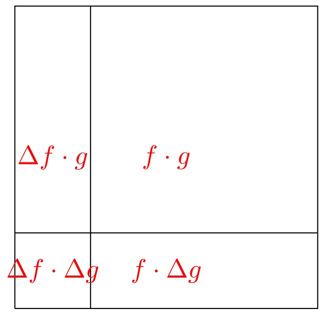
First, let's look at one of our functions.



Draw x^2 or something, put down x and $x+h$. Mark $f(x)$ and $f(x+h)$ as well.

Then let $\Delta f = f(x+h) - f(x)$. Similarly $\Delta g = g(x+h) - g(x)$.

And look at $(f + \Delta f)(g + \Delta g)$.



So then (since we only want the change portion, we remove fg)

$$\Delta(fg) = (f + \Delta f)(g + \Delta g) - fg = \Delta f \cdot g + f \cdot \Delta g + \Delta f \cdot \Delta g$$

In other words,

$$\frac{\Delta(fg)}{h} = g\frac{\Delta f}{h} + f\frac{\Delta g}{h} + \Delta g\frac{\Delta f}{h}$$

So now, let's take limits of all the sides.

$$\begin{aligned}
\lim_{h\to 0}\frac{\Delta(fg)}{h} &= \lim_{h\to 0}\frac{(fg)(x+h) - (fg)(x)}{h} \\
&= (fg)'(x) \\
\lim_{h\to 0}\left(g\frac{\Delta f}{h} + f\frac{\Delta g}{h} + \Delta g\frac{\Delta f}{h}\right) &= g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + f\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\Delta g \cdot \frac{\Delta f}{h} \\
&= g(x)\cdot f'(x) + f(x)\cdot g'(x) + \left(\lim_{h\to 0} g(x+h) - g(x)\right)\cdot f'(x) \\
&= g(x)\cdot f'(x) + f(x)\cdot g'(x) + 0
\end{aligned}$$

In our prime notation we have

$$(fg)' = gf' + fg'$$

This is called the *product rule*: If f and g are both differentiable, then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Example 5.9 Let's do an example. Let $f(x) = x^2e^x$. What is $f'(x)$? First, $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}e^x = e^x$. Then:

$$\frac{df}{dx} = x^2\frac{d}{dx}e^x + e^x\frac{d}{dx}x^2 = x^2e^x + 2xe^x$$

We can also do quotients based off a similar technique as the product.

$$\begin{aligned}
\Delta\left(\frac{f}{g}\right) &= \frac{f+\Delta f}{g+\Delta g} - \frac{f}{g} \\
&= \frac{(f+\Delta f)\,g - f\,(g+\Delta g)}{g\,(g+\Delta g)} \\
&= \frac{fg + g\Delta f - fg - f\Delta g}{g\,(g+\Delta g)} \\
&= \frac{g\Delta f - f\Delta g}{g\,(g+\Delta g)} \\
\lim_{h\to 0}\frac{\Delta\left(\frac{f}{g}\right)}{h} &= \left(\frac{f}{g}\right)' \\
\lim_{h\to 0}\frac{\frac{g\Delta f - f\Delta g}{g(g+\Delta g)}}{h} &= \lim_{h\to 0}\frac{\frac{g\Delta f}{h} - \frac{f\Delta g}{h}}{g\,(g+\Delta g)} \\
&= \frac{g\lim_{h\to 0}\frac{\Delta f}{h} - f\lim_{h\to 0}\frac{\Delta g}{h}}{g\lim_{h\to 0}(g+\Delta g)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}
\end{aligned}$$

In our prime notation we have

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

This is called the *quotient rule*: If f and g are both differentiable and $g(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$$

Note that in the quotient rule, the *order* matters since we are subtracting. Make sure to keep this in mind.

Example 5.10 Let $f(x) = \frac{x^3+3x}{x^2+1}$.

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(3x^2+3) - (x^3+3x)(2x)}{(x^2+1)^2} \\ &= \frac{3x^4+3x^2+3x^2+3-2x^4-6x^2}{x^4+2x^2+1} \\ &= \frac{x^4+3}{x^4+2x^2+1} \end{aligned}$$

Therefore,

$$f'(x) = \frac{x^4+3}{x^4+2x^2+1}$$

Exercise 5.11 With a partner find $f'(x)$ where $f(x) = \underline{\frac{x^4+3x^2+x-1}{e^x}}$.

$$\begin{aligned} f'(x) &= \frac{e^x(4x^3+6x+1) - (x^4+3x^2+x-1)e^x}{(e^x)^2} \\ &= \frac{e^x\left(x^4+4x^3+3x^2+7x\right)}{(e^x)^2} \end{aligned}$$

Example 5.12 Our next real world example comes from electrical engineering. There's a law in EE called "Ohm's Law" which states that the voltage is equal to the current times the resistance at any given point in a circuit,*i.e.*, $V = IR$. We're going to try and calculate the instantaneous voltage at a given time if both the current and the resistance is changing over time.

If $I = i(t)$ is the function for the amount of current at any given time, then the average current over time is given by:

$$\text{average current } = \frac{\Delta I}{\Delta t}$$

and the instantaneous current at any given time is given by:

$$\text{current } = \lim_{\Delta t \to 0} \frac{\Delta I}{\Delta t} = \frac{dI}{dt} = i'(t)$$

Similarly, if $R = r(t)$ is the function for the resistance at any given time, then the average resistance over time is given by:

$$\text{average resistance } = \frac{\Delta R}{\Delta t}$$

and the instantaneous resistance at any given time is given by:

$$\text{resistance } = \lim_{\Delta t \to 0} \frac{\Delta R}{\Delta t} = \frac{dR}{dt} = r'(t)$$

We know that the average rate of voltage over time is given by:

$$\text{average voltage } = \frac{\Delta V}{\Delta t}$$

and, by what we saw earlier, the instantaneous voltage is given by:

$$\text{voltage } = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt} = \underline{i'(t)r(t) + r'(t)i(t)}$$

Let's take as an example $I = i(t) = t^2$ and $R = r(t) = e^t$. Then what is the instantaneous voltage when $t = 2$?

$$\begin{aligned}
i'(t) &= 2t \\
r'(t) &= e^t \\
i'(t)r(t) + r'(t)i(t) &= 2t \cdot e^t + e^t \cdot t^2 \\
&= 2te^t + t^2 e^t \\
&= 2 \cdot 2 \cdot e^2 + 2 \cdot 2 \cdot e^2 \\
&= 4e^2 + 4e^2 \\
&= 8e^2
\end{aligned}$$

Week 6

10–14 Feb 2020

1 Day 1: 10 Feb 2020

6.1 Recall:

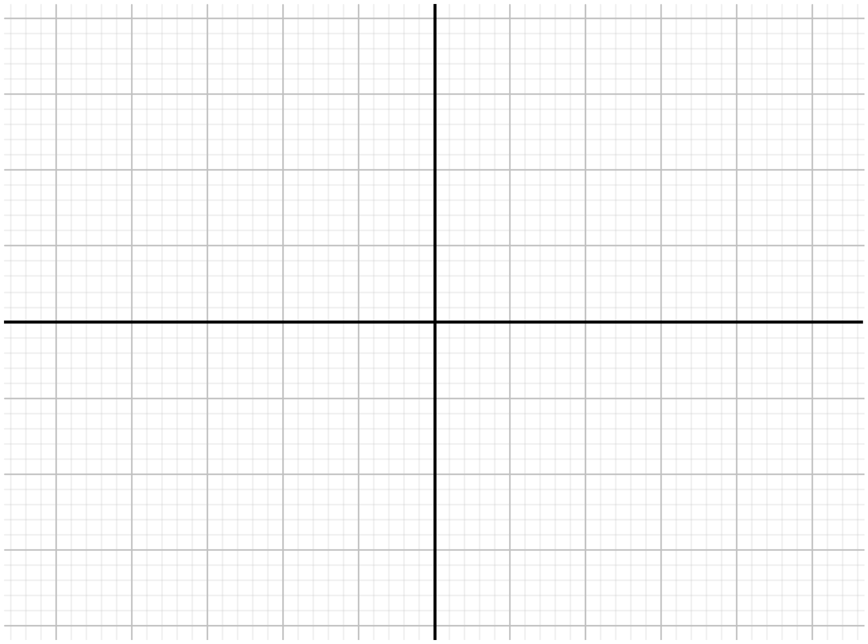
6.2 Derivatives of Trig functions - §3.3

Let's try and find the derivative of $\sin(x)$. By definition we have:

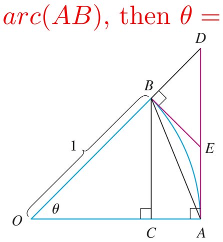
$$\begin{aligned}
f(x) &= \sin(x) \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \qquad \sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a) \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\
&= \lim_{h\to 0} \sin(x)\frac{\cos(h) - 1}{h} + \cos(x)\frac{\sin(h)}{h} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}
\end{aligned}$$

So we need to figure out what these last two limits are to find the derivative. We're going to use the squeeze theorem! First, we want to show that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$ where θ is an angle.

Want to show why $\theta < \tan(\theta)$



Show $\sin(\theta) < \theta = arc(AB)$, then $\theta = arc(AB) < \tan(\theta)$



(a)

Therefore, intuitively, we know that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$. Now, we take the limits of everything as θ goes to 0.

$$\lim_{\theta\to 0} \cos(\theta) = \underline{\cos(0) = 1}.$$

Therefore, by the squeeze theorem $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = \underline{1}$.

Now let's look at $\frac{\cos(\theta)-1}{\theta}$.

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} &= \underline{\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}\cdot\frac{\cos(\theta)+1}{\cos(\theta)+1}} \\
&= \underline{\lim_{\theta\to 0}\frac{\cos^2(\theta)-1}{\theta\left(\cos(\theta)+1\right)} \qquad \sin^2(\theta)+\cos^2(\theta)=1} \\
&= \underline{\lim_{\theta\to 0}\frac{-\sin^2(\theta)}{\theta\left(\cos(\theta)+1\right)}} \\
&= \underline{-\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{\cos(\theta)+1}} \\
&= -\lim_{\theta\to 0}\frac{\sin(\theta)}{\cos(\theta)+1} \\
&= \underline{-\frac{0}{1+1}} \\
&= \underline{0}
\end{aligned}$$

Therefore,

$$f'(x)=\sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}=\underline{\cos(x)}$$

Using very similar methods, we can prove derivatives for all of the trig functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin(x)$	$\underline{\cos(x)}$	$\csc(x)$	$\underline{-\csc(x)\cot(x)}$
$\cos(x)$	$\underline{-\sin(x)}$	$\sec(x)$	$\underline{\sec(x)\tan(x)}$
$\tan(x)$	$\underline{\sec^2(x)}$	$\cot(x)$	$\underline{-\csc^2(x)}$

Example 6.1 Let's use the quotient rule to show one of these. Suppose that $f(x)=\frac{\sin(x)}{\cos(x)}=\tan(x)$ and find $f'(x)$.

$$\begin{aligned}
f'(x) &= \frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x)
\end{aligned}$$

Exercise 6.2 With a partner, find the derivative of $f(x) = \sin(x)\cos(x)$.

$$\begin{aligned} f'(x) &= \sin(x)(-\sin(x)) + \cos(x)\cos(x) \\ &= \cos^2(x) - \sin^2(x) \\ &= \cos(x+x) \\ &= \cos(2x) \end{aligned}$$

6.3 The chain rule - §3.4

We did addition, subtraction, multiplication and division, but there's one operation we haven't touched yet: composition. What is the derivative of $(f \circ g)$?

Suppose we have $f(x)$ and $g(x)$. Normally when we compute the derivative of $f(x)$ we compute $\frac{df}{dx}$ since x is inside of f. In our case, we have $f(g(x))$, so we can think of the derivative as

$$\frac{df}{dg}\frac{dg}{dx}$$

In prime notation, we have

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

This is called the *chain rule*. We won't prove it in class, but it's available in the book if you want to see how it works.

Example 6.3 Let $f(x) = \sqrt{\sin(x)}$. What is $f'(x)$?

$$f(x) = (\sin(x))^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2\sqrt{\sin(x)}} \cdot \cos(x)$$

Exercise 6.4 Let $f(x) = \frac{1}{\sqrt[3]{x^2+3x}}$. With a partner, find $f'(x)$.

$$f(x) = \left(x^2+3x\right)^{\frac{-1}{3}} \qquad f'(x) = \frac{-1}{3}\left(x^2+3x\right)^{\frac{-4}{3}}(2x+3)$$

Let's use the chain rule to find the derivative of an exponential function. Let $f(x) = b^x$, and we want to find $f'(x)$. We're going to use the fact that $b = e^{\ln(b)}$.

$$\ln(b) = \ln(e^{\ln(b)}) = \ln(b)\ln(e)$$

So we have

$$f(x) = b^x = \left(e^{\ln(b)}\right)^x = e^{x\cdot\ln(b)}$$
$$f'(x) = e^{x\cdot\ln(b)}\frac{d}{dx}(x\cdot\ln(b)) = \ln(b)e^{x\cdot\ln(b)} = \ln(b)b^x$$

Example 6.5 Next, we look at an example from chemistry! Suppose we have two substances: substance A and substance B; and when we mix them together we get a substance C. Substances A and B are known as the *reactants* and substance C is known as the *product*. The *concentration* of a reactant or product is normally denoted by the number of moles per liter where a mole is roughly 6.022×10^23 molecules (this number is called *Avogadro's number*). So the concentrations of each substance is given by a function over time. The average concentration of substance C is given by $\frac{\Delta C}{\Delta t}$. As one might expect, the instantaneous concentration at a given time is given by $\frac{dC}{dt}$.

Suppose the concentration of C over a given time is given by the function $c(t) = \cos(2t^2)$. What is the concentration of c at $t = 1$?

So first we need to calculate $c'(t)$.

$$\frac{d}{dt}2t^2 = 4t$$
$$\frac{d}{dt}\cos(t) = -\sin(t)$$
$$c'(t) = -\sin(2t^2)\cdot 4t$$

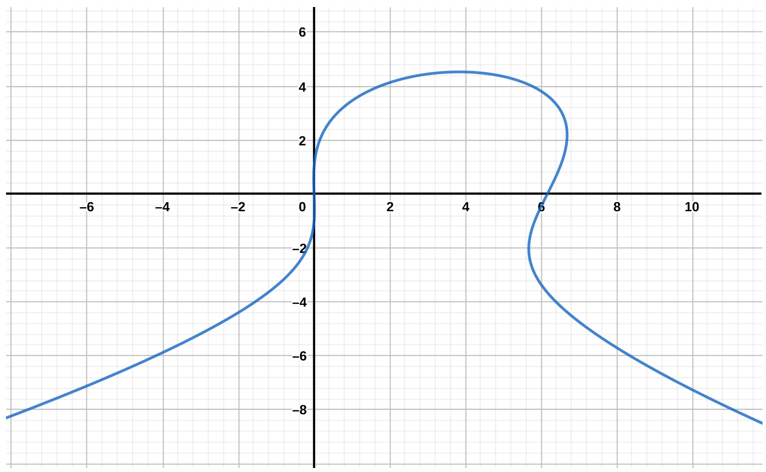
2 Day 2: 12 Feb 2020

6.1 Recall:

6.2 Implicit differentiation - §3.5

So far, we've been looking at functions that have been explicitely defined: $f(x) = \cdots$. But what happens if our function is implicitly defined:

$$y^3 + 6x^2 - 2xy = 37x + y$$



Are there ways to find the tangent to the curve? (aka the derivative) yes!

What we do is we break down our differentiation by small sections. This is best done with an example.

Example 6.6 Let $x^3 + y^3 = 37$. We take the derivative of both sides

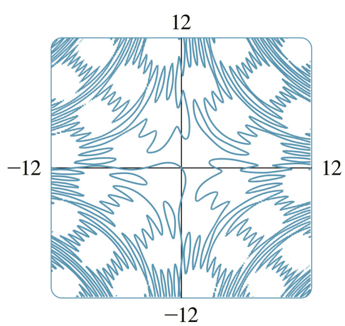
$$\begin{aligned}
\frac{d}{dx}\left(x^3 + y^3\right) &= \frac{d}{dx}37 \\
\frac{d}{dx}x^3 + \frac{d}{dx}y^3 &= \frac{d}{dx}37 \\
3x^2 + \frac{d}{dx}y^3 &= 0 \\
3x^2 + \frac{d}{dy}y^3\frac{dy}{dx} &= 0 \\
3x^2 + 3y^2\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} = y' &= \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}
\end{aligned}$$

Let's try a little more complicated example.

Example 6.7 Let $y^3 + 6x^2 - 2xy = 37x + y$.

$$\begin{aligned}
\frac{d}{dx}\left(y^3 + 6x^2 - 2xy\right) &= \frac{d}{dx}\left(37x + y\right) \\
\frac{d}{dx}y^3 + \frac{d}{dx}6x^2 - \frac{d}{dx}2xy &= \frac{d}{dx}37x + \frac{d}{dx}y \\
\frac{d}{dy}y^3\frac{dy}{dx} + 12x - (2y + 2x\frac{d}{dx}y) &= 37 + \frac{d}{dy}y\frac{dy}{dx} \\
3y^2\frac{dy}{dx} + 12x - 2y - 2x\frac{d}{dy}y\frac{dy}{dx} &= 37 + \frac{dy}{dx} \\
3y^2\frac{dy}{dx} + 12x - 2y - 2x\frac{dy}{dx} &= 37 + \frac{dy}{dx} \\
3y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - \frac{dy}{dx} &= 37 + 2y - 12x \\
\frac{dy}{dx} &= \frac{37 + 2y - 12x}{3y^2 - 2x - 1}
\end{aligned}$$

Exercise 6.8 Let $\sin(xy) = \sin(x) + \sin(y)$.



With a partner, find y'.

$$\begin{aligned}
\sin(xy) &= \sin(x) + \sin(y) \\
\cos(xy)(y + xy') &= \cos(x) + \cos(y)y' \\
\cos(xy)y - \cos(x) &= \cos(y)y' - \cos(xy)xy' \\
y' &= \frac{\cos(xy)y - \cos(x)}{\cos(y) - \cos(xy)x}
\end{aligned}$$

Example 6.9 Let $x^3+y^3 = 37$. Find y''. We saw earlier that $y' = \frac{dy}{dx} = \frac{-x^2}{y^2}$.

$$\begin{aligned}
y'' = \frac{d}{dx}y' = \frac{d}{dx}\left(\frac{-x^2}{y^2}\right) \\
&= -\frac{y^2\frac{d}{dx}x^2 - x^2\frac{d}{dx}y^2}{(y^2)^2} \\
&= -\frac{y^2 \cdot 2x - x^2\frac{d}{dy}y^2\frac{dy}{dx}}{y^4} \\
&= -\frac{2xy^2 - x^2 \cdot 2y \cdot \frac{-x^2}{y^2}}{y^4} \\
&= -\frac{2xy^2 + \frac{2x^4}{y}}{y^4} \\
&= -\frac{2xy^3 + 2x^4}{y^5}
\end{aligned}$$

6.3 Derivatives of inverse trig functions - §3.5

We can use implicit differentiation in order to find the derivatives of inverse trig functions. Recall that $\arcsin(x) = \sin^{-1}(x) = y$ means that $\sin(y) = x$ when $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$. So let's test it out.

$$\begin{aligned}
\frac{d}{dx}\sin(y) &= \frac{d}{dx}x \\
\frac{d}{dy}\sin(y)\frac{dy}{dx} &= 1 \\
\cos(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos(y)}
\end{aligned}$$

Since $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ then $\cos(y) \geq 0$. We can see this through circle graph/grid of sin/cos. And since $\sin^2(y) + \cos^2(y) = 1$ we have

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \underline{\sqrt{1 - x^2}}.$$

In other words

$$\frac{d}{dx}\arcsin(x) = \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

Using similar approaches, we can find the derivatives for every inverse trig function:

$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arccsc}(x)$	$-\frac{1}{x\sqrt{x^2-1}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcsec}(x)$	$\frac{1}{x\sqrt{x^2-1}}$
$\arctan(x)$	$\frac{1}{1+x^2}$	$\operatorname{arccot}(x)$	$-\frac{1}{1+x^2}$

3 Day 3: 14 Feb 2020

6.1 Recall:

6.2 Derivatives of logarithm functions - §3.6

What is the derivative of the function $f(x) = \log_b(x)$?

Recall that $y = \log_b(x)$ means that $b^y = x$. So let's implicit differentiate!

$$\begin{aligned}
\frac{d}{dx}b^y &= \frac{d}{dx}x \\
\underline{\frac{d}{dy}b^y\frac{dy}{dx}} &= \underline{1} \\
\underline{\ln(b)b^y\frac{dy}{dx}} &= \underline{1} \\
\frac{dy}{dx} &= \underline{\frac{1}{\ln(b)b^y} = \frac{1}{\ln(b)x}}
\end{aligned}$$

The derivative comes from Exercise 6.3.

What this means is that if $b = e$ then

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Example 6.10 Let $f(x) = \ln(\cos(x))$, what is $f'(x)$?

$$f'(x) = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$

Example 6.11 Let $f(x) = \left(\ln\left(\frac{x^3}{x^2+1}\right)\right)^2$, what is $f'(x)$?

$$f'(x) = 2\left(\ln\left(\frac{x^3}{x^2+1}\right)\right)\cdot\frac{1}{\frac{x^3}{x^2+1}}\cdot\frac{(x^2+1)(3x^2)-x^3(2x)}{(x^2+1)^2} = \frac{2\ln\left(\frac{x^3}{x^2+1}\right)(x^2+3)}{x(x^2+1)}$$

Exercise 6.12 Let $f(x) = x^2\ln\left(\frac{x+3}{x^2}\right)$, with a partner, find $f'(x)$.

$$\begin{aligned} f'(x) &= 2x\ln\left(\frac{x+3}{x^2}\right) + \frac{x^2}{\left(\frac{x+3}{x^2}\right)}\cdot\frac{x^2\cdot 1-(x+3)\cdot 2x}{(x^2)^2} \\ &= 2x\ln\left(\frac{x+3}{x^2}\right) - \frac{x^2+6}{x+3} \end{aligned}$$

The logarithm can actually help us differentiate more complicated rational functions. This is best demonstrated with an example.

Example 6.13 Let

$$f(x) = \frac{x^{\frac{9}{4}}(x^3-1)^{\frac{4}{3}}}{(7x^2-1)^4}$$

and find $f'(x)$. This equation is super complicated and we can use the chain rule, the product rule and the quotient rule to solve it, but that's going to get very complicated very fast. Instead, we can use logarithms to simplify everything. We do this in 3 steps.

(1) Take logs of both sides and simplify. Let $y = f(x)$. Then

$$\ln(y) = \ln\left(\frac{x^{\frac{9}{4}}(x^3-1)^{\frac{4}{3}}}{(7x^2-1)^4}\right) = \frac{9}{4}\ln(x) + \frac{4}{3}\ln\left(x^3-1\right) - 4\ln\left(7x^2-1\right)$$

(2) Differentiate implicitly with respect to x.

$$\begin{aligned} \frac{d}{dx}\ln(y) &= \frac{d}{dx}\left(\frac{9}{4}\ln(x) + \frac{4}{3}\ln\left(x^3-1\right) - 4\ln\left(7x^2-1\right)\right) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{9}{4x} + \frac{4x^2}{(x^3-1)} - \frac{56x}{7x^2-1} \end{aligned}$$

(3) Solve for y' .

$$\begin{aligned}\frac{dy}{dx} &= y\left(\frac{9}{4x} + \frac{4x^2}{(x^3-1)} - \frac{56x}{7x^2-1}\right)\\ &= \left(\frac{x^{\frac{9}{4}}(x^3-1)^{\frac{4}{3}}}{(7x^2-1)^4}\right)\left(\frac{9}{4x} + \frac{x^2}{(x^3-1)} - \frac{56x}{7x^2-1}\right)\end{aligned}$$

Example 6.14 Let's try it again with a super simple example. Let $f(x) = x^x$.

(1) Let $y = f(x)$. Then

$$\ln(y) = \ln(x^x) = x\ln(x)$$

(2)

$$\begin{aligned}\frac{d}{dx}\ln(y) &= \frac{d}{dx}x\ln(x)\\ \frac{1}{y}\frac{dy}{dx} &= \frac{x}{x} + \ln(x)\end{aligned}$$

(3)

$$\begin{aligned}\frac{dy}{dx} &= y\left(1+\ln(x)\right)\\ &= x^x\left(1+\ln(x)\right)\end{aligned}$$

Exercise 6.15 Let

$$f(x) = \frac{x^{\frac{4}{3}}\left(x^3 - 3x^2\right)^{\frac{7}{3}}}{\sqrt{x^2+1}}$$

with a partner, find $f'(x)$.

(1)

$$\ln(y) = \frac{4}{3}\ln(y) + \frac{7}{3}\left(x^3 - 3x^2\right) - \frac{1}{2}\ln(x^2+1)$$

(2)

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{3x} + \frac{7(x-2)}{x^2-3x} - \frac{x}{x^2+1}$$

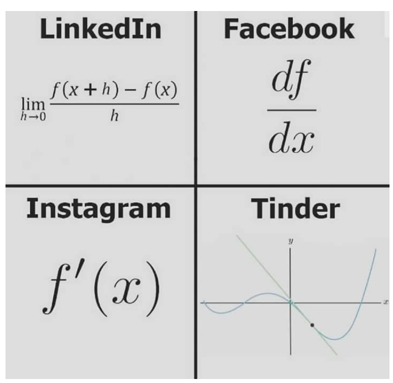
(3)

$$f'(x) = \frac{x^{4/3}\left(x^3-3x^2\right)^{7/3}}{\sqrt{x^2+1}} \cdot \left(\frac{4}{3x} + \frac{7(x-2)}{x^2-3x} - \frac{x}{x^2+1}\right)$$

Week 7

Reading Week - 17–21 Feb 2020

No class this week. Instead take a break, catch up on this class and all your other classes. There are no office hours this week. Here's a meme to help you feel better.



Source: twitter - @GrumpyReviewer2

Bonus! Here's your chance for some extra credit! 10 points extra credit (on the next exam) if you can create an original meme based on what we've learned in this class. Due this Friday! (21 Feb)

Note that it must be of your own creation. If I find the meme online/-somewhere else, then you will lose 2 points on your next exam instead.

Week 8

24–28 Feb 2020

1 Day 1: 24 Feb 2020

8.1 Recall:

8.2 Rates of Change - §3.7

In the last couple weeks we've been looking at different ways to apply derivatives into the real world. Most of these came through rates of change.

What is "rate of change"? It means "how much something is changing". This can be interpreted in many different ways! In the astronaut example, we talked about how velocity is the rate of change of position. We talked about acceleration being the rate of change of velocity. In a lot of our other examplese we looked at how much something changes over time and figuring out the instantaneous rate of change at some fixed time. We looked at the following:

- Acceleration/Velocity (Astronauts!)
- Density of objects
- Electricity flow (current and Ohm's law)
- Rates of chemical reactions
- Population growth

We're now going to continue looking at how derivatives help in the real world.

The examples here are the same examples in the book. There are a bunch of even more exciting examples that can be found through a quick google search of applications of derivatives.

Example 8.1 We first look at thermodynamics. In thermodynamics we're looking at how mcuh something can be compressed. So given some amount of pressure, the volume of a substance is going to change. The rate of change is then just the change of volume over pressure: dV/dP. The *isothermal compressibility* is then defined to be:

$$\text{isothermal compressibility } = \beta = -\frac{1}{V}\frac{dV}{dP}$$

In other words, β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases (at constant temperature).

Let $V = \frac{3.5}{P^2}$ and let's figure out the isothermal compressibility when the pressure is 10 kPa (kilopascals). First we must calculate dV/dP.

$$\frac{dV}{dP} = \underline{-\frac{7}{P^3}}$$

Then we calculate β.

$$\beta = -\frac{1}{V}\frac{dV}{dP} = \underline{-\frac{1}{\frac{3.5}{P^2}}\left(-\frac{7}{P^3}\right)} = \frac{7P^2}{3.5P^3} = \frac{2}{P}$$

So then $\beta(10) = \underline{0.2}$.

Example 8.2 Our next example looks at economics. If we let $C(x)$ denote the total cost that a company incurs in producing x number of products then C is called the *cost function* of x. What happens if we change the number of products produced? This gives us a rate of change! We get the following:

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$$

which tells us how much everything is changing. The *marginal cost* is just the instantaneous rate of change of the cost. In other words

$$\text{marginal cost } = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

Wait!!! Hold up. You can't have half a product. Like if I'm selling a doll, I can't just sell half a doll! So what does $\Delta x \to 0$ even mean?! Basically what we are saying here is that if the number of products is big enough then we can appoximate the addition of one additional product, by taking the derivative. In other words we are saying:

$$C'(n) \approx C(n+1) - C(n)$$

This makes life easier since maybe calculating C twice is much more difficult for a computer (or a human) than calculating the derivative once and calculating it once.

Let's pretend we are working for Willy Wonka and he wants to figure out the marginal cost of the ever-lasting gobstopper. He's super secretive so he only wants to produce 10 of them for now. What is Wonka's marginal cost if $C(x) = \operatorname{arcsec}(x) + x^2$.

First we must find the derivative!

$$C'(x) = \frac{1}{x\sqrt{x^2-1}} + 2x$$

Then calculating at $x = 10$ we get:

$$C'(10) = \frac{1}{10 \cdot \sqrt{10^2 - 1}} + 2 \cdot 10 = \frac{1}{10\sqrt{99} + 20}$$

8.3 Exponential Growth and Decay - §3.8

We've also done some exponential growth and decay problems already. We saw this in the population growth case.

A lot of times, things grow and decay proportional to themselves. So in essence, in a population, you grow based on how many people are present, and that growth in measured by some proportion of the current number. As an example: If we take a census in 2015 and then take a census in 2020 then we might find that the population grew by 20%. What this means is that if P is the population in 2015, then $1.2P$ is the population in 2020. So the population is measured in proportions.

This rate of change can be made instantaneous by taking the derivative, giving us:

$$\frac{dy}{dt} = ky$$

where k is the proportion.

We've only seen one function where y stays itself after taking the derivative! The exponential!!! That means, usually with growth and decay, we are dealing with the exponential function.

In fact, it is known that the only solutions to the equation $\frac{dy}{dt}ky$ are the exponential functions:

$$y(t) = y(0)e^{kt}$$

Let's look at some examples where we use these exponential functions.

Example 8.3 Let's first look at the nucleur armageddon. In other words: radioactive decay. This example is different from the population growth one because we are actually decreasing the volume of a radioactive substance rather than increasing. In essence, if we have some particle then it's mass decays exponentially by the following formula:

$$m(t) = m_0 e^{kt}$$

where m_0 is the initial value, k is some constant for the particle, and t is the variable. In order to calculate this, physicists use something called the half-life of a particle. The *half-life* of a particle is the time needed for half of the particle to decay. We use all these facts in the following example.

So let's suppose we have $1kg$ of uranium-235. It takes 704 million years for 1 gram of uranium-235 to decay to 1/2 a gram. In other words, the half-life of uranium-235 is 704 million years = 704000000. We already know that we are starting off with 1000 grams of uranium-235 so that means $m_0 = 1000$. But how do we find k?

Using the half-life! We already know that $m(704000000)$ should be half the initial mass since 704000000 is the half-life. So

$$\begin{aligned} m(704000000) &= m_0 e^{kt} \\ &= 1000e^{k704000000} \\ 500 &= 1000e^{k704000000} \\ \frac{1}{2} &= e^{k704000000} \\ \ln(\frac{1}{2}) &= 704000000k \\ -\ln(2) &= 704000000k \\ k &= \frac{-\ln(2)}{704000000} \end{aligned}$$

So

$$m(t) = \underline{m_0 e^{-\ln(2)t/704000000}} = m_0 2^{-t/704000000}$$

How much of our uranium will be left over in 5 billion years when the sun dies out?

$$m(5000000000) = 1000 \cdot 2^{-5000000000/704000000} = 1000 \cdot 2^{-5000/704} = \frac{1000}{2^{625/88}} \approx 7.3$$

Example 8.4 Our next example uses Newton's law of cooling. We're going to be looking at the temperature of an object over some time, so we denote this function by $T(t)$. Newton's law of cooling says that

$$\frac{dT}{dt} = k(T - T_s)$$

where T_s is the temperature of the surroundings. To calculate things, we need to fix things up. So we let $y(t) = T(t) - T_s$. Then $y'(t) = T'(t)$ and so our equation above becomes:

$$y'(t) = ky$$

which matches our exponential formula from above.

So suppose that we just made some hot boiling tea (or coffee) and it's just chilaxing at room temperature (20°). We know that in 30 minutes, the tea will be 55°. Considering that the perfect drinking temperature for tea is 65°, when should you drink your tea?

This is a ton of information! So let's break it down slow. First, let's figure out our formula:

$$\frac{dT}{dt} = k(T - T_s) = \underline{k(T - 20)}$$

We also know that $T(0) = \underline{100}$ and so $y(0) = T(0) - T_s = \underline{100 - 20 = 80}$. Also, we know that $T(30) = \underline{55}$ and so $y(30) = T(30) - T_s = \underline{55 - 20 = 35}$.

Then $y(t) = y(0)e^{kt}$ (from our equation earlier!). We can now calculate k:

$$\begin{aligned} y(t) = 35 &= y(0)e^{kt} \\ &= 80e^{k \cdot 30} \\ \frac{35}{80} &= e^{30k} \\ \ln(\frac{7}{16}) &= 30k \\ \frac{\ln(\frac{7}{16})}{30} &= k \end{aligned}$$

So now we want to figure out when $T(t) = \underline{65}$ in other words when $y(t) = \underline{65 - 20 = 45}$. Therefore:

$$\begin{aligned} y(t) = 45 &= y(0)e^{kt} \\ 45 &= 80e^{kt} \\ \ln(\frac{45}{80}) &= kt \\ \ln(\frac{9}{16}) &= \frac{\ln(7/16)}{30}t \\ t &= 30\frac{\ln(9/16)}{\ln(7/16)} \\ &\approx 30\frac{-0.57364}{-0.82668} \\ &\approx 20.88 \end{aligned}$$

In other words, we must wait about 20 minutes after pouring tea before it is at perfect temperature.

2 Day 2: 26 Feb 2020

8.1 Recall:

8.2 Related Rates - §3.9

Related rates are notoriously complicated. They are word problems that are difficult to understand and don't relate to anything in the real world. Although they are not fun, they teach us how to problem solve in the real world. At a job, no one is ever going to give you a function and say "find its derivative". Instead they are going to explain a problem to you, give you a ton of data, and expect you to solve it. This is what this section aims to do:

(1) We first explain what the situation is.

(2) Then we give a bunch of data.

(3) Finally, you figure out how to solve it.

These are hard because they require a lot more thinking than just memorization. Even though they are difficult, there are some steps you can take to help you:

(1) Read the problem carefully. (1 minute rule)

(2) Draw a diagram if possible. (pictures help!)

(3) Introduce notation. (symbols, functions, etc.)

(4) Write an equation relating things. (use tools from before this class to simplify)

(5) Differentiate to get the appropriate rates.

(6) Substitute the given information into the resulting equation.

(7) Solve (and be set free!)

Example 8.5 You're walking along the street one day when and you remember it's your friend's birthday tomorrow! You totally forgot to by them a birthday present!!! But then you come up with a brilliant idea, you're going to buy them a ton of balloons with their face on each one. You decide to go to Sky's: your local non-binary balloon seller! After a 30 minute walk, you get to Sky's and ask them about making you balloons with your friend's picture on it. Sky turns to you and says they would love to help, but their boss came up with a new rule. All employees need to be able to calculate the rate at which the radius of a balloon increases or else they can't sell any balloons. They turn to you and ask if you would help them figure it out. Armed with calculus, you tell Sky yes, and buckle down to help them.

Sky tells you that when the put the balloon on the helium pump, the spherical balloon's volume increases at 100 cm^3/s. Can you help them figure out the rate at which the radius of the baloon is increasing? What's the rate when the diameter is equal to 20 cm?

(1) Highlight:

   (a) **Explain:** "Need to calculate the rate at which radius of a balloon increases"

   (b) **Data:** "Spherical", "Volume increases at", "Diameter is "

   (c) **Solve:**

(2) Draw a sphere to calculate volume.

(3) Introduce notation: V-volume; r-radius; d-diameter

(4) Write an equation relating things: $V = \frac{4}{3}\pi r^3$; $d = 2r$.

$$\frac{dV}{dt} = 100\ cm^3/s \qquad \frac{dr}{dt} = ?$$

(5) Differentiate to get appropriate rates

$$\begin{aligned}\frac{d}{dt}V &= \frac{dV}{dt}\frac{4}{3}\pi r^3\\ \frac{dV}{dt} &= \frac{dV}{dr}\frac{4}{3}\pi r^3\frac{dr}{dt}\\ &= 4\pi r^2\frac{dr}{dt}\\ \frac{dr}{dt} &= \frac{1}{4\pi r^2}\frac{dV}{dt}\end{aligned}$$

(6) Subtitute given information:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} 100$$

But we also know r! Since $d = 20$ and $d = 2r$ therefore $r = 10$.

$$\frac{dr}{dt} = \frac{1}{4\pi(10)^2} 100$$

(7) Solve!

$$\frac{dr}{dt} = \frac{1}{4\pi 100} 100 = \frac{1}{4\pi}$$

Example 8.6 After grabbing your balloons you continue on your way home. As you're heading home you decide to walk by a construction site to see your favorite construction worker: Sandra. Did you know, only 4% of construction workers are female?! You walk over to the construction site and see Sandra standing next to a ladder on the ground getting ready to do something. Wanting to know what's on her mind, you grab a hard hat from the table and walk over to her to see what's up. Not looking away from the ladder, she tells you she was just about to move the ladder to start painting the wall. She then turns to you and notices all your balloons and a grin appears on her face! "We should tie the balloons to the ladder and let the balloons lift the ladder up", she says. Although you don't want to lose the balloons, this seems like fun so you go with it. She takes the balloons and puts them on the 10 metre long ladder. You both start noticing that the ladder starts moving away from you both at a rate of 1 metres per second. How fast will the top of the ladder be moving away from you when the ladder is 6 metres up?

(1) Highlight:

  (a) **Explain:** "How fast will the top of the ladder be moving"

  (b) **Data:** "6 metres up", "10 metres long", "1 metres per second, "on the ground"

  (c) **Solve:**

(2) Draw a triangle.

(3) Introduce notation: y is distance from corner to top ladder, x is distance from corner to bottom ladder,

(4) Write an equation relating things: $x^2 + y^2 = 10^2 = 100$ by triangle rules!

$$\frac{dx}{dt} = -1 \qquad \frac{dy}{dt} = ?$$

(5) Differentiate to get appropriate rates

$$\begin{aligned}\frac{d}{dt}(x^2+y^2) &= \frac{d}{dt}100\\ \frac{d}{dt}x^2+\frac{d}{dt}y^2 &= 0\\ 2x\frac{dx}{dt}+2y\frac{dy}{dt} &= 0\\ \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt}\end{aligned}$$

(6) Subtitute given information:

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{x}{6}(-1)$$

But we also can calulate x!

$$x^2 = 10^2 - y^2 = 100 - 36 = 64 = 8^2$$

$$\frac{dy}{dt} = -\frac{x}{6}(-1) = -\frac{8}{6}(-1)$$

(7) Solve!

$$\frac{dy}{dt} = \frac{4}{3}$$

Exercise 8.7 After helping Sandra with her ladder, you realize you're gonna be late to your dinner date tonight! You run home, drop off the balloons and quickly freshen up. Looking at your watch you realize you're defs going to be late if you don't leave stat. You call an uber and decide to click on the "sports car driver" option. Within 30 seconds a sports car driver meets you in front of your house, introduces himself as Vito. Remembering that Vito was your city's four time reigning champion in sports car driving, you know you're gonna make it to your date on time. You hop in his car and you're on your way.

As you're heading to the restaurant, you look at Vito's speedometer and notice he's going 100 km/h! And in a school area! You're confident you'll make it in time, only to realize you're coming up to the "intersection of doom". You look around and notice another sports car uber driver on the other street approaching the intersection as well. The other driver is Morty! The infamous sports car driver from the city over who won nationals last year! Realizing that if your car doesn't hit the intersection first, you're going to be late to your date, you decide to quickly calculate who's going to arrive at the intersection first. Although you are 4 km from the intersection, you notice that the other car is going arond 80 km/h and that the distance between your cars is decreasing by 80 km/h. Will you make it to your date in time?

(1) Highlight:

   (a) **Explain:** "Which car gets to the intersection first?"

   (b) **Data:** "100 km/h", "80 km/h", "4 km", "80 km/h",

   (c) **Solve:**

(2) Draw a triangle.

(3) Introduce notation: x is distance to intersection for your uber driver, y is distance to intersection for the other car, z is distance between cars.

(4) Write an equation relating things: $x^2 + y^2 = z^2$ by triangle rules!

$$y = ?$$

(5) Differentiate to get appropriate rates

$$\begin{aligned}
\frac{d}{dt}(x^2+y^2) &= \frac{d}{dt}z^2 \\
\frac{d}{dt}x^2 + \frac{d}{dt}y^2 &= \frac{d}{dt}z^2 \\
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2z\frac{dz}{dt} \\
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2\sqrt{x^2+y^2}\frac{dz}{dt}
\end{aligned}$$

(6) Subtitute given information:

$$2(4)(100) + 2y(80) = 2\sqrt{3^2+y^2}80$$

(7) Solve!

$$\begin{aligned}
800 + 160y &= 160\sqrt{9+y^2} \\
\frac{800}{160} + y &= \sqrt{9+y^2} \\
\frac{80}{16} + y &= \sqrt{9+y^2} \\
\frac{40}{8} + y &= \sqrt{9+y^2} \\
\frac{20}{4} + y &= \sqrt{9+y^2} \\
5 + y &= \sqrt{9+y^2} \\
(5+y^2) &= 9+y^2 \\
25 + 10y + y^2 &= 9 + y^2 \\
10y &= 9 - 25 = -16 \\
y &= \frac{-8}{5}
\end{aligned}$$

Example 8.8 Although you were late for your date, your date doesn't mind and you end up having an amazing time at the restaurant. So good in fact, that you invite them to your friend's birthday party tomorrow! When tomorrow comes, you decide you need to make a cake for your friend, but you want it to be a special cake, so you're going to make it a cone shaped cake! You look for your cone shaped pan and find the one with a 10 cm radius and is 20 cm tall. You make the cake mix and start pouring the batter into the pan (pointy side down) at around 2 cm^3/min. While your pouring, your date from last night calls and asks what you're doing before the party. You tell them and they ask you how fast the batter is rising in the pan? You notice that your batter just hit the 5 cm mark, how fast is it rising?

(1) Highlight:

  (a) **Explain:** "How fast is the batter rising?"

  (b) **Data:** "Find the one with", "pointy side down", "at around 2 cm^3/min".

  (c) **Solve:**

(2) Draw a cone

(3) Introduce notation: h is height, r is radius. V is volume.

(4) Write an equation relating things: $V = \frac{1}{3}\pi r^2 h$. Similar triangles $\frac{r}{h} = \frac{1}{2}$.

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

(5) Differentiate to get appropriate rates

$$\frac{d}{dt}V = \frac{d}{dt}\frac{\pi h^3}{12}$$
$$\frac{dV}{dt} = \frac{3\pi h^2}{12}\frac{dh}{dt}$$

(6) Subtitute given information:

$$(2) = \frac{3\pi(5)^2}{12}\frac{dh}{dt}$$

(7) Solve!

$$2 = \frac{3\pi 5^2}{12}\frac{dh}{dt}$$
$$2 = \frac{25\pi}{4}\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{8}{25\pi}$$

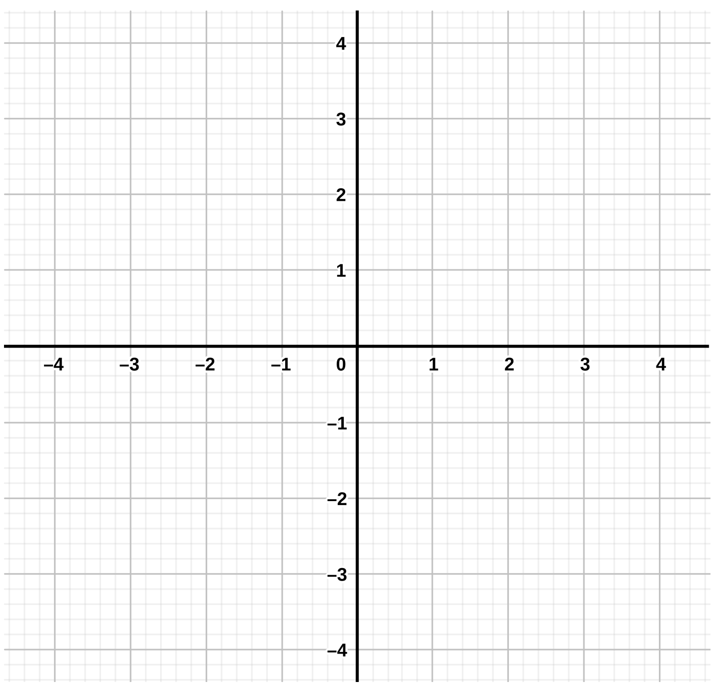
3 Day 3: 28 Feb 2020

8.1 Recall:

8.2 Linear approximations and differentials - §3.10

Recall that the derivative gives us the slope of the tangent line to a function. What if we want to get an approximation of the line itself and not just the slope?

Let a be some point on a graph.



We want an equation of the form $y = mx + b$. We already know the slope! The slope is given by its derivative: $f'(a)$. We only need to figure out what b is.

To find b, we start at $f(a)$ and we "remove" the part "rise" part of our slope formula. So recall slope is $\frac{\text{rise}}{\text{run}}$. So

$$f'(a) = \frac{\text{rise}}{\text{run}} = \frac{\text{rise}}{a}$$

So we have $b = f(a) - f'(a)a$.

So our approximation for the line is given by:

$$y = f'(a)x + f(a) - f'(a)a = f(a) + f'(a)(x - a).$$

The approximation $f(x) \approx f(a) + f'(a)(x - a)$ is called the *linear approximation* or *tangent line approximation* of f. The linear function whose graph is the tangent line is given by

$$L(x, a) = f(a) + f'(a)(x - a)$$

is called the *linearization of f at a*.

The book uses $L(x)$ instead of $L(x, a)$. This is confusing because the linearization depends on our choice of a! We'll see this in the next example.

Example 8.9 Let $f(x) = \underline{x^2 - 3x + 1}$. Find the linearization of f at 1 and at 3.

First, let's find $f'(x)$.

$$f'(x) = 2x - 3$$

(1) Let $a = 1$. Then $f(a) = 1^2 - 3 \cdot 1 + 1 = 1 - 3 + 1 = -1$ and $f'(a) = 2 \cdot 1 - 3 = 2 - 3 = -1$. Therefore,

$$L(x, 1) = f(1) + f'(1)(x - 1) = -1 + (-1)(x - 1) = -1 - x + 1 = -x$$

(2) Let $a = 3$. Then $f(a) = 3^2 - 3 \cdot 3 + 1 = 9 - 9 + 1 = 1$ and $f'(a) =$

$2 \cdot 3 - 3 = 6 - 3 = 3$. Therefore,

$$L(x, 3) = f(3) + f'(3)(x - 3) = 1 + 3(x - 3) = 1 + 3x - 9 = 3x - 8$$

Notice that these two linearizations are different! It's important to remember that a linearization depends on your choice of a.

How close is this approximation though?

Example 8.10 Suppose we want to calculate $f(3.112347)$ from the function above. In a calculator, we find:

$$f(3.112347) = 1.34966285$$

but sometimes, life happens and we don't have access to a calculator and/or the calculator is wrong. So let's use our linear approximation to see what the approximation would give. We said that $f(x) \approx L(x, a)$ for some a close to x. Let $a = 3$ since 3 and 3.112347 are fairly "close". Then

$$L(3.112347, 3) = \underline{3(3.112347) - 8} = 9.337041 - 8 = 1.337041$$

This is not to bad of an approximation!

The whole point of approximation is to make the calculation significantly easier by using a linear function to approximate, rather than trying to solve some big nasty complicated formula.

Example 8.11 Find a good approximation for $f(x) = x^x$ for $x = \frac{8}{9}$. Let's first find $L(x, a)$ for an a close to $\frac{8}{9}$. Let $a = 1$ and let's solve.

$$f(1) = 1^1 = 1$$

Recall that $f'(x) = x^x(1 + \ln(x))$ so

$$f'(1) = 1(1 + \ln(1)) = 1(1 + 0 + 1$$

Then

$$L(x, 1) = f(1) + f'(1)(x - 1) = 1 + 1(x - 1) = x$$

So $f(\frac{8}{9}) \approx L(\frac{8}{9}, 1) = \frac{8}{9}$. In fact,

$$\frac{8}{9} = 0.88888\ldots \qquad \left(\frac{8}{9}\right)^{\frac{8}{9}} \approx 0.9005982343\ldots$$

which is not horrific.

Exercise 8.12 With a partner, find the linearization of $f(x) = \sin(x)$ for x really close to 0.

$$f(0) = \sin(0) = 0 f'(0) \quad = \cos(0) = 1 L(x,0) = f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$$

Week 9

2–6 Mar 2020

1 Day 1: 2 March 2020

9.1 Recall:

9.2 Differentials - §3.10

Remember how we had said that $\frac{dy}{dx}$ is not a fraction and we shouldn't think of it as one? We're about to change all of that!

We can think of dy as a very small change, an approximation of a larger Δy. Let's just pretend that $f'(x) = \frac{dy}{dx}$ is a fraction and put like terms on the same side. So we have $\underline{dy = f'(x)dx}$. We call dx and y <u>*differentials*</u> .

It turns out we can use differentials as a good approximation of Δy. Let's see an example to see what in the world I'm talking about.

Example 9.1 Let $f(x) = x^2 - 2x + 4$. Let us look at the change from 2 to 2.01.

Recall

$$\Delta y = f(x + \Delta x) - f(x) \text{ and } x = 2,\ \Delta x = 0.01$$

Then

$$f(2) = 2^2 - 2 \cdot 2 + 4 = 4 \quad f(2.01) = (2.01)^2 - 2.01 \cdot 2 + 4 = 4.0401 - 4.02 + 4 = 4.0201$$

So $\Delta y = 4.0201 - 4 = 0.0201$

Now, $f'(x) = 2x - 2$. So

$$f'(2) = 2 \cdot 2 - 2 = 2$$

and $dx \approx \Delta x = 0.01$. Then

$$dy = f'(2)dx = 2 \cdot 0.01 = 0.02$$

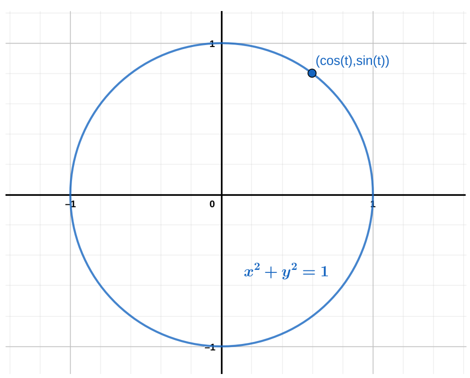
Notice that

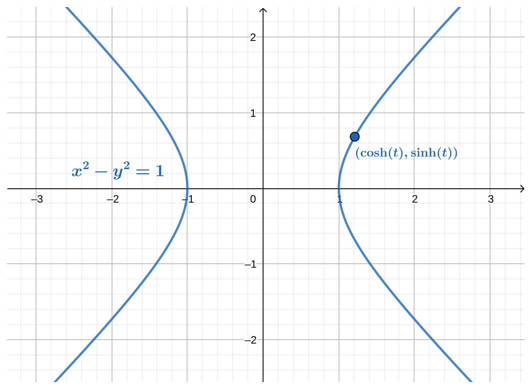
$$dy = 0.02 \approx 0.0201 = \Delta y$$

So this "division" turns out to be a good approximation! Also, finding dy was a lot easier to compute than Δy.

9.3 Hyperbolic functions - §3.11

Let's talk about hyperpbolic functions next. Recall that we can get the sin and cos functions from a circle. If instead we looked at a hyperbola, we would get the hyperbolic functions.

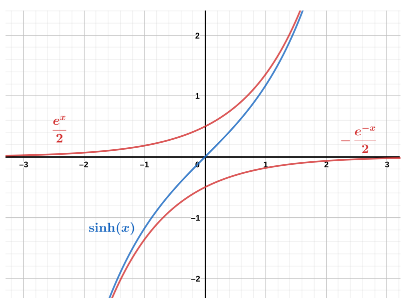


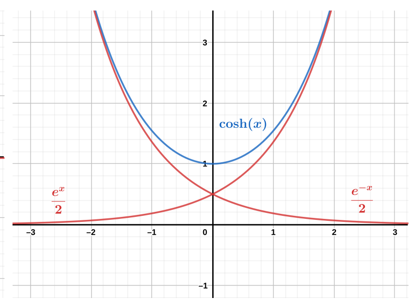


They're hard to remember, so here are their definitions:

$\sinh(x)$	$\frac{e^x - e^{-x}}{2}$	$\operatorname{csch}(x)$	$\frac{1}{\sinh(x)}$
$\cosh(x)$	$\frac{e^x + e^{-x}}{2}$	$\operatorname{sech}(x)$	$\frac{1}{\cosh(x)}$
$\tanh(x)$	$\frac{\sinh(x)}{\cosh(x)}$	$\coth(x)$	$\frac{1}{\tanh(x)}$

Here is what $\sinh(x)$ and $\cosh(x)$ look like





If we're lucky, we'll see some applications of these functions later in the term, but for now let's look at some of the properties of these functions.

$$\begin{aligned} \sinh(-x) &= -\sinh(x) & \cosh(-x) &= \cosh(x) \\ \sinh(x+y) &= \sinh(x)\cosh(x) + \cosh(x)\sinh(y) & \cosh^2(x) - \sinh^2(x) &= 1 \\ \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) & 1 - \tanh^2(x) &= \operatorname{sech}^2(x) \end{aligned}$$

We can also compute the derivatives of the hyperbolic functions fairly easily using their original definitions using exponential functions.

Example 9.2 Let $f(x) = \cosh(x)$, what is its derivative?

$$\begin{aligned}\frac{d}{dx}\cosh(x) &= \frac{d}{dx}\frac{e^x + e^{-x}}{2}\\ &= \frac{1}{2}\frac{d}{dx}\left(e^x + e^{-x}\right)\\ &= \frac{1}{2}\left(e^x - e^{-x}\right)\\ &= \sinh(x)\end{aligned}$$

Here are all the derivatives for the hyperbolic functions.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sinh(x)$	$\cosh(x)$	$\operatorname{csch}(x)$	$-\operatorname{csch}(x)\coth(x)$
$\cosh(x)$	$\sinh(x)$	$\operatorname{sech}(x)$	$-\operatorname{sech}(x)\tanh(x)$
$\tanh(x)$	$\operatorname{sech}^2(x)$	$\coth(x)$	$-\operatorname{csch}^2(x)$

Just like the trig functions, hyperbolic functions have inverses too.

$f(x)$	$f^{-1}(x)$	Domain
$\sinh(x)$	$\operatorname{arsinh}(x) = \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$	$\mathbb{R}$
$\cosh(x)$	$\operatorname{arcosh}(x) = \cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$	$[1, \infty)$
$\tanh(x)$	$\operatorname{artanh}(x) = \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$
$\operatorname{csch}(x)$	$\operatorname{arcsch}(x) = \operatorname{csch}^{-1}(x) = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right)$	$\mathbb{R}\setminus\{0\}$
$\operatorname{sech}(x)$	$\operatorname{arsech}(x) = \operatorname{sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0, 1]$
$\coth(x)$	$\operatorname{arcoth}(x) = \coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$

Similarly to the inverse trig functions, the inverse hyperbolic functions have derivatives.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\operatorname{arsinh}(x)$	$\frac{1}{\sqrt{1+x^2}}$	$\operatorname{arcsch}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$\operatorname{arcosh}(x)$	$\frac{1}{\sqrt{x^2-1}}$	$\operatorname{arsech}(x)$	$-\frac{1}{x\sqrt{1-x^2}}$
$\operatorname{artanh}(x)$	$\frac{1}{1-x^2}$	$\operatorname{arcoth}(x)$	$\frac{1}{1-x^2}$

$\operatorname{artanh}(x)$ and $\operatorname{arcoth}(x)$ appear to have the same function as a derivative, but recall that $\operatorname{artanh}(x)$ and $\operatorname{arcoth}(x)$ have different domains, so the derivatives are defined on different parts of the graph.

2 Day 2: 4 Mar 2020

9.1 Recall:

9.2 Maximum and minimum values - §4.1

A lot of problems in life are optimization problems: when do functions reach maximum/minimum values. We kinda saw this with our astronaut problem where we wanted to find maximum acceleration. We also see this for example when trying to decide:

- What was the peak value of CO_2 in the atmosphere before 0 BCE?
- What time do restaurants have the most amount of clientele?
- What shape of packaging for our product do we need to minimize costs?

There are an endless supply of max/min problems.

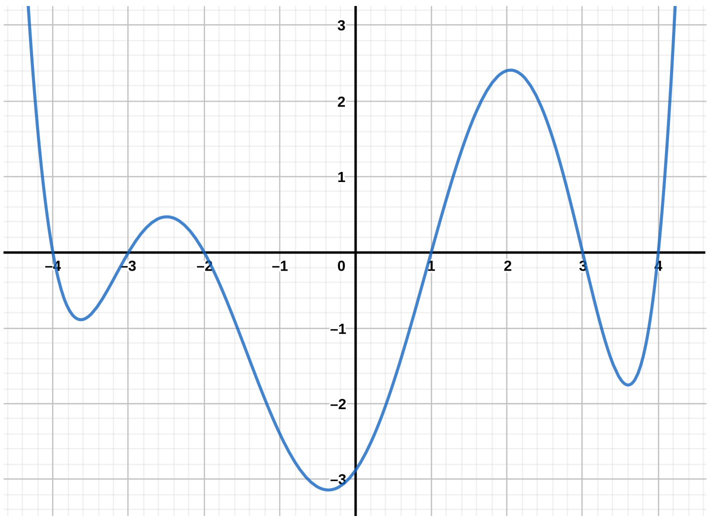
Exercise 9.3 Try and come up with a max/min problem and share it with your neighbor:

Max/Min problem: Let them write something here.

Mathematically, what is maximum and minimum though? A function f has an *absolute maximum value at c on a domain D* if $f(c) \geq f(x)$ for every $x \in D$. Similarly, a function f has an *absolute minimum value at c on a domain d* if $f(c) \leq f(x)$ for every $x \in D$. Absolute values are sometimes known as global values. An *extreme value of f* is an absolute maximum or minimum value of f.

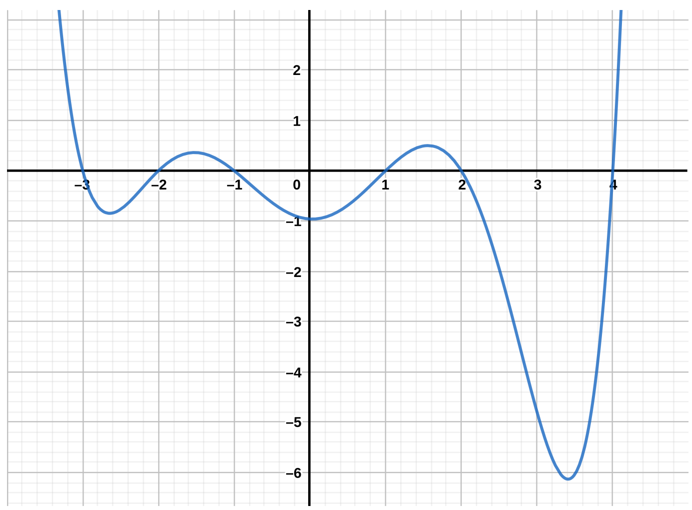
We can also look at points which are maximal/minimal in a smaller area. If $f(c) \geq f(x)$ for every x near c then we say $f(c)$ is a *local maximum*. Similarly, $f(c)$ is a *local minimum* if $f(c) \leq f(x)$ for every x near c.

Example 9.4



Restrict to $[-4, 4]$ and give abs max/min, local max/min

Exercise 9.5 With a partner, fill in the absolute max/min and the local max/mins in the domain $[-3, 4]$ on the following graph:



How do we know that extreme values even exist? Maybe we have some graph where there are no extreme values! This turns out not to be the case as was shown by Bernard Bolzano in the 1830s when he proved the following result.

Theorem 9.6 (Extreme value theorem) *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

What about local maxima and minima? It turns out that Pierre Fermat gave a nice property on local maxima and minima.

Theorem 9.7 (Fermat's Theorem) *If f has a local maximum or minimum at c and if $f'(c)$ exists then $f'(c) = 0$.*

The book (rightfully so) puts up a million warnings about this theorem. It's very easy to want to say that the theorem says more than it actually does. There are 3 common errors that students make with this theorem, two of which are mentioned in the book.

(1) Just because $f'(c) = 0$ that does not mean the function has a local max/min at c. If $f(x) = x^3$, then $f'(0) = 0$, but the graph shows we don't have a local max/min.

(2) Just because $f(x)$ has a local max/min at a point doesn't mean we can just let $f'(x) = 0$ to find the point. If $f(x) = |x|$ then $f'(0)$ does not exist even though it is a local min. So we can't state that $f'(0) = 0$.

(3) Just because $f(x)$ has a local max/min and $f'(c) = 0$ for some c does not mean that the function rises to c and then falls back down. If $f(x) = a$ for some constant a, then $f'(x) = 0$ for all x. Notice that we have a local max/min at every point and that there is no "hump" in our function.

Although there are a lot of easy pitfalls with Fermat's theorem, it does suggest that *maybe*, we can determine how a function looks based on its derivative. A *critical number* of a function f is a number c (in the domain) such that $f'(c) = 0$ or $f'(c)$ does not exist. In other words, Fermat's theorem becomes:

If f has a local maximum or minimum at c then c is a critical number of f.

It turns out we can use cricital numbers to tell us when we have an absolute max or min in some interval $[a, b]$. For this we:

(1) Find the critical values of f in the interval (a, b).

(2) Find the values of f at the critical values and at the end points of the interval (at a and b).

(3) The largest value of the previous step is the absolute maximum and the smallest value is the absolute minimum.

Example 9.8 Let $f(x) = x^3 - x^2 - x$, find its absolute maximum and minimum on the interval $[\frac{-1}{2}, 2]$.

(1) First we find critical points.

$$f'(x) = 3x^2 - 2x - 1 = (3x+1)(x-1)$$

Therefore, critical points at $x = 1$ and $x = \frac{-1}{3}$, both of which are in our interval.

(2) Solve at end points and at critical points.

$$\begin{aligned} f(1) &= 1 - 1 - 1 = -1 \\ f(\frac{-1}{3}) &= \frac{-1}{27} - \frac{1}{9} + \frac{1}{3} = \frac{5}{27} \\ f(\frac{-1}{2}) &= \frac{-1}{8} - \frac{1}{4} + \frac{1}{2} = \frac{1}{8} \\ f(2) &= 8 - 4 - 2 = 2 \end{aligned}$$

(3) So we have an absolute maximum value of 2 and an absolute minimum value at -1.

Exercise 9.9 Let $f(x) = x^3 + x^2 - 1$, find its extreme values on the interval $[-1, 1]$.

(1) Critical pairs

$$f'(x) = 3x^2 + 2x = x(3x+2)$$

Therefore $x = 0$ and $x = \frac{-2}{3}$.

(2) Solve at end points and at critical points.

$$\begin{aligned} f(0) &= -1 \\ f(1) &= 1 \\ f(-1) &= -1 \\ f(-\frac{2}{3}) &= \frac{-23}{27} \end{aligned}$$

(3) Max: 1, Min: -1.

3 Day 3:6 Mar 2020

9.1 Recall:

9.2 The mean value theorem - §4.2

Our next goal is the mean value theorem. We start off with Rolle's theorem, which was named after Michel Rolle who published it in 1691 for polynomial functions. Although Rolle was the first to publish his ideas, Indian mathematician Bhāskara II (1114–1185) is credited to be the first person we know of to have knowledge of the theorem. In addition, the theorem itself was not proved in full until 1823 when Cauchy proved it in its entirety.

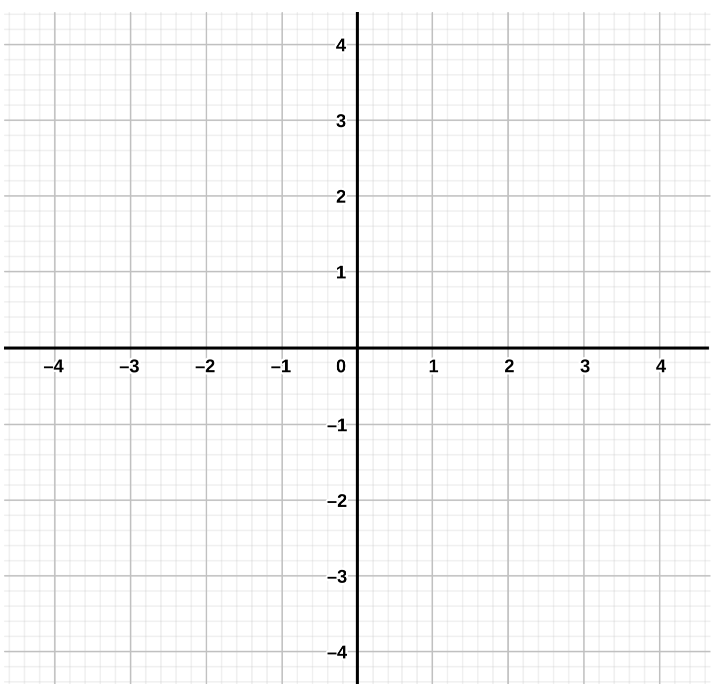
Theorem 9.10 (Rolle's theorem) *Let f be a function that satisfies the following three hypotheses:*

(1) f is continuous on $[a, b]$.

(2) f is differentiable on (a, b).

(3) $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.



Show this theorem by example.

Rolle's theorem is used to find the number of roots a function might have.

Example 9.11 Show that the equation $f(x) = x^3$ has exactly one real root. First notice that f is continuous and differentiable on the entire real line, so we can use Rolle's theorem whenever we have an interval $[a, b]$ where $f(a) = f(b)$. We know that there is at least one real root since $f(-1) = -1$ and $f(1) = 1$. So by the intermediate value theorem, at least one real root exists. In fact, $f(0) = 0$ is this real root.

Now suppose there is a second real root at a point c. Then $f(0) = f(c) = 0$ and by Rolle's theorem there is a point d inside the interval $(0, c)$ such that $f'(d) = 0$. Solving, we get $f'(d) = 3d^2 = 0$ is only true when $d = 0$. But 0 is not contained in the open interval $(0, c)$, therefore we have a contradiction and there is no second real root.

The mean value theorem was known first to Parameshvara (1370–1460), an Indian mathematician. In Europe, it was first stated by Joseph-Louis Lagrange in the following form.

Theorem 9.12 (Mean value theorem) *Let f be a function such that:*

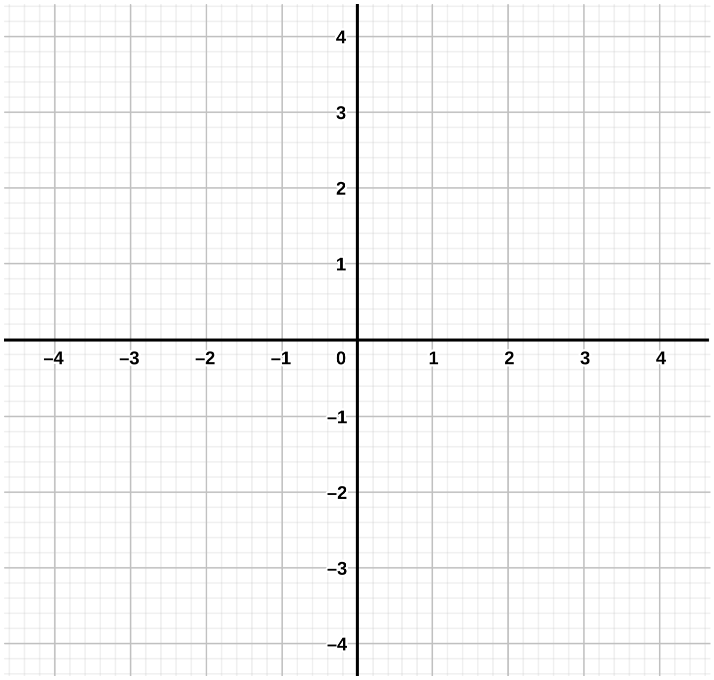
(1) f is continuous on $[a, b]$.

(2) f is differentiable on (a, b).

Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Sometimes this theorem is known as *Lagrange's mean value theorem*. There is an extended version of this theorem which we won't go over called the *extended mean value theorem* or *Cauchy's mean value theorem*.



Show this theorem by example.

Example 9.13 Let $f(x) = x^2 - 2$ and let's look at the interval $[0, 2]$. Show the mean value thorem is true on this interval. We already know f is continuous on $[0, 2]$ and differentiable on $(0, 2)$ since f is a polynomial. Then

$$\frac{f(2) - f(0)}{2 - 0} = \frac{2 - (-2)}{2 - 0} = \frac{4}{2} = 2$$

and

$$f'(c) = 2c$$

Thereofre

$$2c = 2 \Rightarrow c = 1$$

The mean value theorem allows us to state the following two theorems.

Theorem 9.14 *If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).*

Corollary 9.15 *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); i.e., $f(x) = g(x) + c$ for some constant c.*

Week 10

9–13 Mar 2020

1 Day 1: 9 Mar 2020

10.1 Recall:

10.2 Derivatives and graphing - §4.3

Our next goal is to look at what the derivative can inform us about the original function. Since the derivative is the slope of the tangent line, when the slope is positive we know the original function is increasing. Likewise, if the slope is negative it's decreasing.

- If $f'(x) > 0$ for all x in an interval, then f is increasing on that interval.
- If $f'(x) < 0$ for all x in an interval, then f is decreasing on that interval.

Example 10.1 Let $f(x) = x^4 + 4x^3 + 4x^2 + 4$. Find which intervals the function is increasing and decreasing.

First we find the derivative.

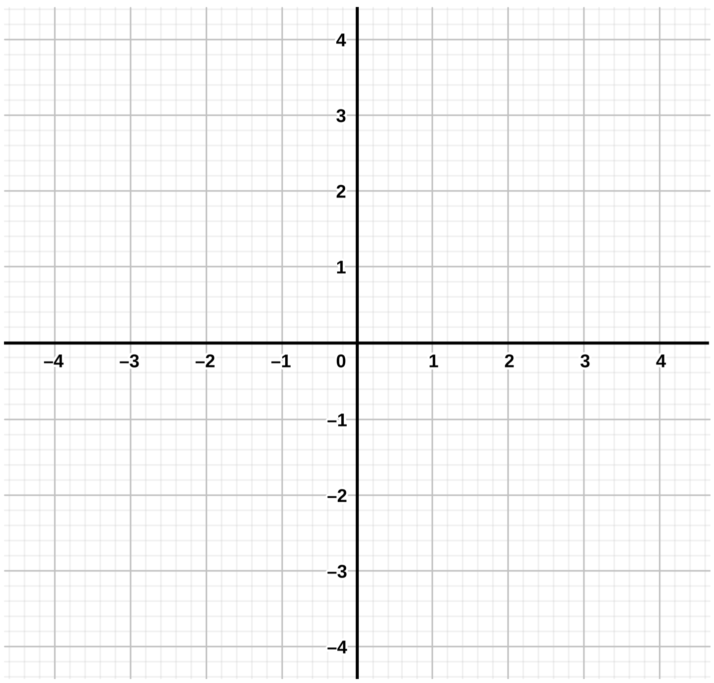
$$f'(x) = 4x^3 + 12x^2 + 8x = 4(x^3 + 3x^2 + 2x) = 4(x)(x+1)(x+2)$$

Then we look at all the ciritical points (in this case $0, 1, 2$).

Let's make a chart:

Interval	x	$x+1$	$x+2$	$f'(x)$	$f(x)$
$x < -2$	$-$	$-$	$-$	$-$	decreasing
$-2 < x < -1$	$-$	$-$	$+$	$-$	increasing
$-1 < x < 0$	$-$	$+$	$+$	$-$	decreasing
$0 < x$	$+$	$+$	$+$	$+$	increasing

Ok, but how can we use this information to actually draw the graph?



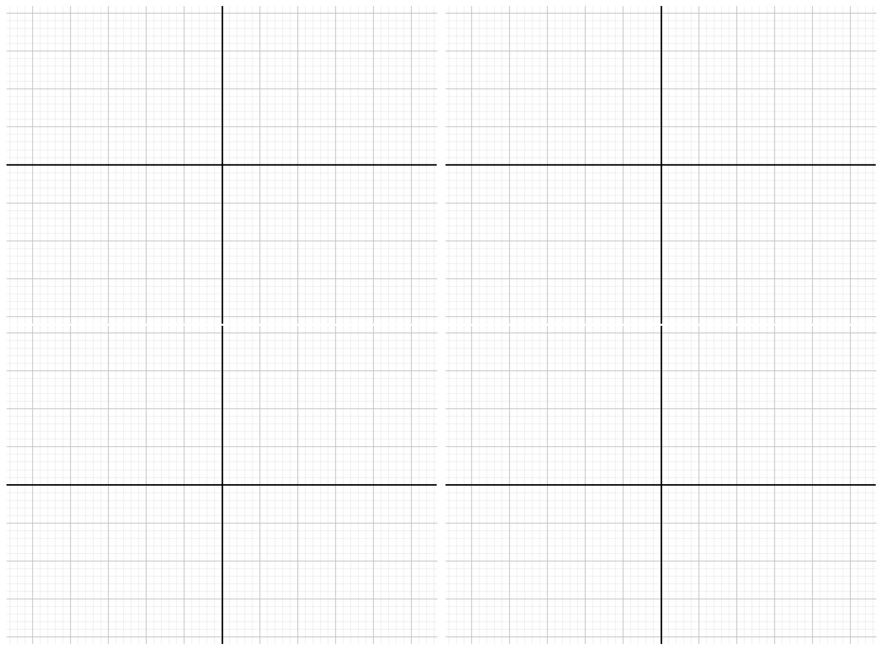
solve $f(x)$ at crit points, then estimate

Notice that this tells us a lot about the function! We can summarize this in the following test.

Theorem 10.2 (The first derivative test) *Let c be a critical number of a continuous function f.*

- *If f' changes from positive to negative at c, then f has a local maximum at c.*
- *If f' changes from negative to positive at c, then f has a local minimum at c.*

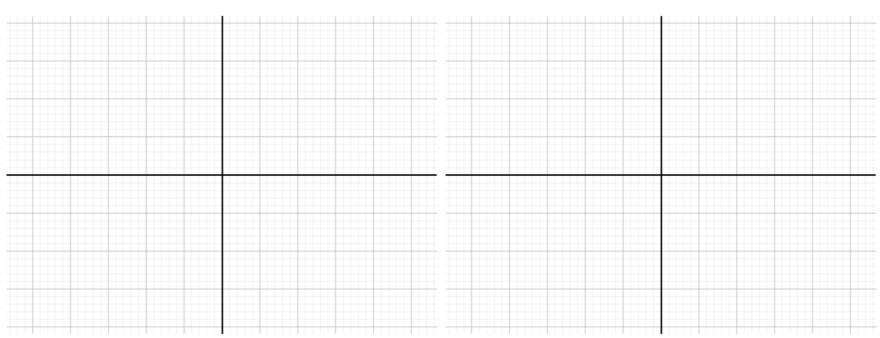
- *If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.*



Show one of each option in a grid

Can we say even more if we look at f''? Yes! It turns out the second derivative helps us distinguish "how" a function is increasing/decreasing. In fact, it tells us about its concavity .

Definition 10.3 If the graph of f lies above all of its tangents on an interval I, then it is called *concave upward* on I. If the graph of f lies below all of its tangents on an interval I, then it is called *concave downward* on I.



Show concave up and down

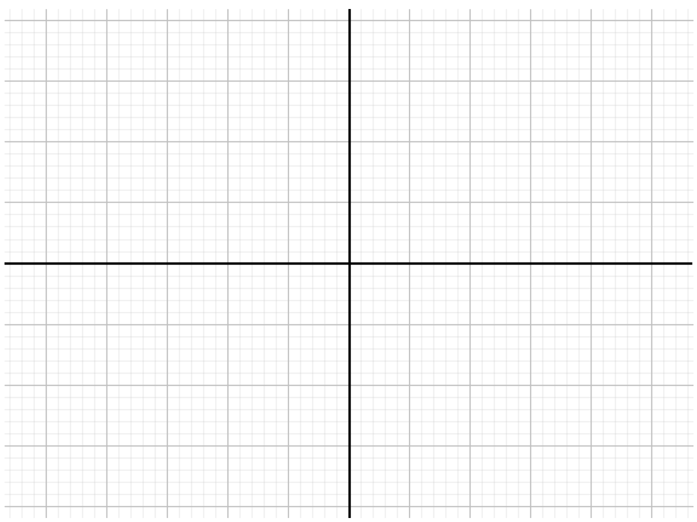
Explain how second derivative relates to concavity. All this boils down to the *concavity test:*

- If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ for all x in an interval I, then the graph of f is concave downward on I.

Sometimes our functions will switch from concave upward to downward or the other way around. The points where this switch occurs is called an *inflection point*.

Let's look at an example.

Example 10.4 Try and sketch the graph of $f(x) = x^6-6x^4-4x^3+9x^2+12x+6$.

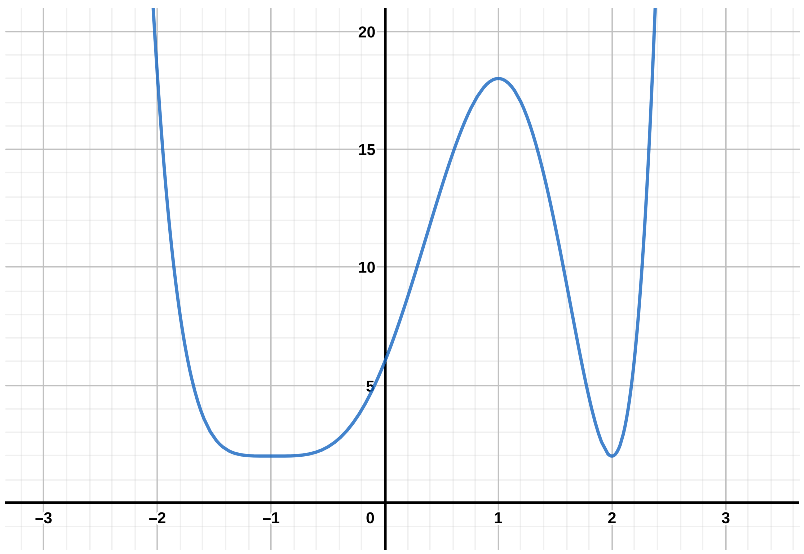


$f(x) = (x^6-6x^4-4x^3+9x^2+12x+6)$ $f'(x) = 6(x^5-4x^3-2x^2+3x+2) = 6(x+1)^3(x-1)(x-2)$ $f''(x) = 6(5x^4-$

Note: $5x^2 - 10x + 3$ has roots between 0 and 1, and between 1 and 2. Say these roots are s and t.

$$f(-1) = 2,\ f(1) = 18,\ f(2) = 2,\ f(s) \approx 11.3,\ f(t) \approx 9.5$$

Interval	$x+1$	$x-1$	$x-2$	$5x^2-10x+3$	$f'(x)$	$f''(x)$	$f(x)$
$x < -1$	$-$	$-$	$-$	$+$	$-$	$+$	decreasing, concave up
$-1 < x < s$	$+$	$-$	$-$	$+$	$+$	$+$	increasing, concave up
$s < x < 1$	$+$	$-$	$-$	$-$	$+$	$-$	increasing, concave down
$1 < x < t$	$+$	$+$	$-$	$-$	$-$	$-$	decreasing, concave down
$t < x < 2$	$+$	$+$	$-$	$+$	$-$	$+$	decreasing, concave up
$2 < x$	$+$	$+$	$+$	$+$	$+$	$+$	increasing, concave up

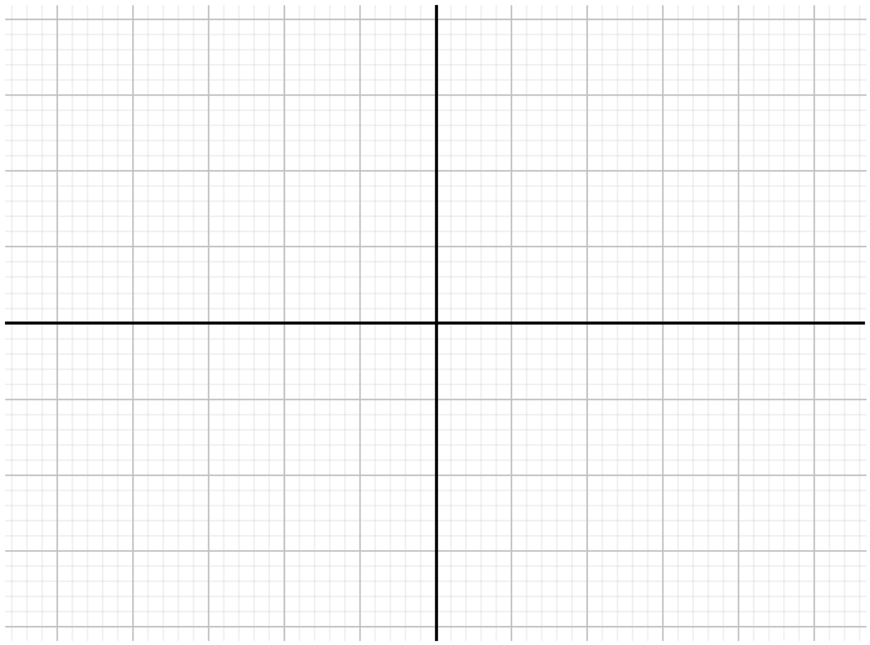


Looking at the second derivative, we can state our derivative tests in a much more concise way.

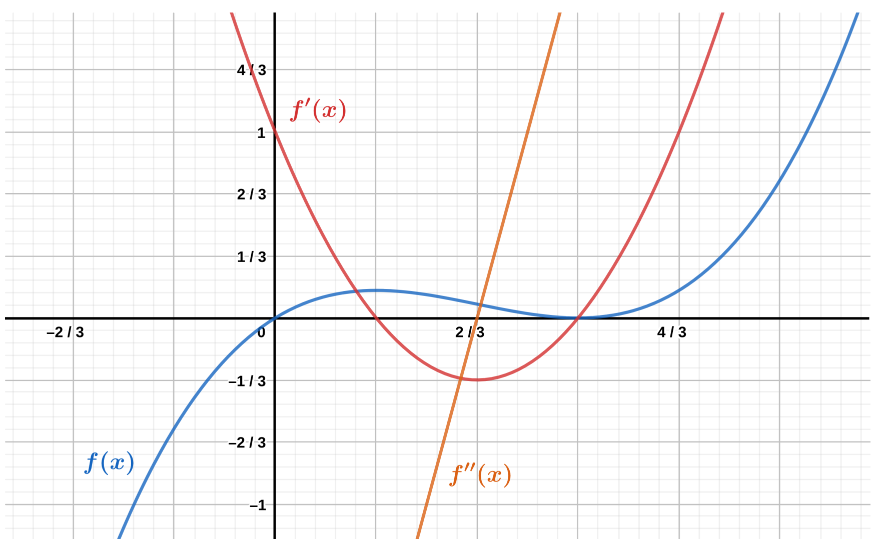
Theorem 10.5 (Second derivative test) *Suppose f'' is continuous near c.*

- *If $f'(c) = 0$ and $f''(c) > 0$, then f ahas a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$, then f ahas a local maximum at c.*

Exercise 10.6 With a partner, sketch the graph of $f(x) = \underline{x(x-1)^2}$.



$f'(x) = (3x - 1)(x - 1) \quad f''(x) = (6x - 4)$



2 Day 2:

10.1 Recall:

10.2 L'Hôpital's Rule

We're going to go back to limits for a minute and look at how we can find limits at places that aren't that well defined. For example, what's the following limit:

$$\lim_{x\to 0} \frac{\sin(x)}{\ln(x)} = \underline{\frac{0}{0} = 0}$$

Or how about

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \underline{\frac{0}{0} = 0}$$

Whenever we have a limit

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \underline{\frac{0}{0} \text{or} \pm \frac{\infty}{\infty}}$$

then this limit is called an <u>*indeterminate form of type* $\frac{0}{0}$ *or* $\pm\frac{\infty}{\infty}$</u> .

These limits aren't very easy to solve by the methods we have so far, but in 1694 the Swiss mathematician Johann Bernoulli introduced a theorem to the French mathematician Guillaume de l'Hôpital which we now call L'hôpital's rule:

Theorem 10.7 (L'Hôpital's rule) *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose further that either*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

(in other words, the limit is an indeterminate form of either type). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \underline{\lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

if the limit on the right side exists or is $\pm\infty$.

This is awesome! Let's try it with our examples from before.

Example 10.8 Let $f(x) = \sin(x)$ and $g(x) = \ln(x)$ and, from above, recall

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{0}{\underline{0}}$$

so we can use L'hôpitals rule!

$$f'(x) = \cos(x) \quad g'(x) = \frac{1}{x}$$

Then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \cos(x)x = 0$$

Example 10.9 Let $f(x) = x^2$ and $g(x) = e^x$ and, from above, recall that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

We can therefore use L'Hôpital's rule!

$$\lim_{x\to 0} \frac{x^2}{e^x} = \lim_{x\to 0} \frac{2x}{e^x} = \lim_{x\to 0} \frac{2}{e^x} = 0$$

Try another example with a partner.

Exercise 10.10 Let $f(x) = x^2 - x - 2$ and $g(x) = \sqrt{2} - \sqrt{x}$. With a partner, find

$$\lim_{x\to 2} \frac{f(x)}{g(x)}$$

$$f'(x) = 2x - 1 \quad g'(x) = \frac{-1}{2\sqrt{x}}$$

$$\lim_{x\to 2} \frac{2x-1}{\frac{-1}{2\sqrt{x}}} = \lim_{x\to 2} \frac{(2x-1)(2\sqrt{2})}{-1} = (2\cdot 2 - 1)2\sqrt{2}\cdot -1 = -6\sqrt{2}$$

Example 10.11 Which of the following can I use L'hôpital's rule on?

$$\lim_{x\to\infty} \frac{1}{x} \qquad \lim_{x\to\infty} \frac{x^2+7x}{x^2+9x+3} \qquad \lim_{x\to\infty} \frac{x^2+3x+1}{(x^4+7x)\cdot e^x}$$

$$\lim_{x\to 0} \frac{e^x}{x} \qquad \lim_{x\to \pi} \frac{\pi - x}{\cos(x)} \qquad \lim_{x\to 0} \frac{0}{\cos(x)}$$

Should I use L'hôpital's rule?

We can also use L'Hôpital's rule in other cases! In most of these cases, what we want to do is convert our limit into a limit of one of our two indeterminate types. These are best done with examples.

The form $\infty - \infty$

Example 10.12 Find the limit

$$\lim_{x\to 0} \frac{1}{x\sin(x)} - \frac{1}{x^2} = \infty - \infty$$

$$\begin{aligned}
\lim_{x\to 0} \tfrac{1}{x\sin(x)} - \tfrac{1}{x^2} &= \lim_{x\to 0} \tfrac{x-\sin(x)}{x^2\sin(x)} = \tfrac{0}{0} \\
&= \lim_{x\to 0} \tfrac{1-\cos(x)}{2x\sin(x)+x^2\cos(x)} = \tfrac{0}{0} \\
&= \lim_{x\to 0} \tfrac{\sin(x)}{2\sin(x)+2x\cos(x)+2x\cos(x)-x^2\sin(x)} = \tfrac{0}{0} \\
&= \lim_{x\to 0} \tfrac{\cos(x)}{2\cos(x)-4x\sin(x)+4\cos(x)-2x\sin(x)-x^2\cos(x)} \\
&= \tfrac{1}{6}
\end{aligned}$$

Notice how we changed our formula from $\infty - \infty$ to $\frac{0}{0}$ in order to be able to use L'Hôpital's rule. We do the same for all the types.

The form $0 \cdot \pm\infty$

Example 10.13 Find the limit

$$\lim_{x\to 0} 8x^4\ln(3x) = 0\cdot\infty$$

$$\begin{aligned}
\lim_{x\to 0} 8x^4\ln(3x) &= \lim_{x\to 0} \tfrac{\ln(3x)}{1/8x^4} = \tfrac{-\infty}{\infty} \\
&= \lim_{x\to 0} \tfrac{3/3x}{-4/8x^3} \\
&= \lim_{x\to 0} \tfrac{2x^3}{-x} \\
&= \lim_{x\to 0} -2x^2 \\
&= 0
\end{aligned}$$

We can do the same with powers!

The forms 0^0, ∞^0, and $1^{\pm\infty}$

In these cases, it's a little more complicated.

Example 10.14 Find the limit

$$\lim_{x\to 0}(x^2)^{x^2} = 0^0$$

$$\begin{aligned}
\lim_{x\to 0}(x^2)^{x^2} &\to \ln\left(\lim_{x\to 0}(x^2)^{x^2}\right)\\
\ln\left(\lim_{x\to 0}(x^2)^{x^2}\right) &= \lim_{x\to 0}\ln\left((x^2)^{x^2}\right)\\
&= \lim_{x\to 0} 2x^2\ln(x) = 0\cdot -\infty\\
&= \lim_{x\to 0}\frac{2\ln(x)}{1/x^2} = \frac{-\infty}{\infty}\\
&= \lim_{x\to 0}\frac{2/x}{-2/x^3}\\
&= \lim_{x\to 0}\frac{2x^3}{-2x}\\
&= \lim_{x\to 0} -x^2 = 0\\
\ln\left(\lim_{x\to 0}(x^2)^{x^2}\right) &= 0\\
&\Rightarrow\\
\lim_{x\to 0}(x^2)^{x^2} &= e^0 = 1
\end{aligned}$$

In essence, we followed the following steps:

(1) Set $L = \lim_{x\to a} f(x)$.

(2) Find $\ln(L) = \ln(\lim_{x\to a} f(x)) = \lim_{x\to a}\ln(f(x)) = N$.

(3) Then $\lim_{x\to a} f(x) = e^N = L$.

Exercise 10.15 With a partner, find the following limit.

$$\lim_{x\to 0}\sin(x)^x$$

$$\begin{aligned}
\ln\left(\lim_{x\to 0}\sin(x)^x\right) &= \lim_{x\to 0} x\ln(\sin(x))\\
&= \lim_{x\to 0}\frac{\ln(\sin(x))}{1/x} = \frac{-\infty}{\infty}\\
&= \lim_{x\to 0}\frac{\cos(x)/\sin(x)}{-1/x^2}\\
&= \lim_{x\to 0}\frac{-x^2\cos(x)}{\sin(x)} = \frac{0}{0}\\
&= \lim_{x\to 0}\frac{-2x\cos(x)+x^2\sin(x)}{\cos(x)} = 0
\end{aligned}$$

Then $e^0 = \lim_{x\to 0} \sin(x)^x = 1$.

3 Day 3:

10.1 Recall:

10.2 Optimization problems - §4.7

Optimization problems are difficult. Not only because they use derivatives, but because they are generally word problems that are difficult to interpret. They are best done *slowly* and with a lot of thought to make sure that you are doing things correctly.

There are 6 basic steps to handling an optimization problem.

(1) Understand the problem

(2) Draw a diagram

(3) Introduce notation

(4) Express desired value in terms of other things

(5) Find relationships between variables so that eqn is in one variable

(6) Find absolute max/min

This is definitely best done with an example.

Example 10.16 Suppose we time travelled to the 8th century and got stuck in the countryside of some kingdom. Looking at our time machine we realize it's going to take up a few years until we can fix it, so we gotta make due and set-up a little farm to sustain ourselves for a little bit. After constructing a cabin that's not too shabby you realize you should probably turn the extra material into a fence to protect your crops from animals. You have material to make a 400 meter fence (you're known for super speed when it comes to chopping wood and making fences) and you can't get more material because. . . life. To help make your plot bigger you decide that one side of the rectangular plot of land will be shielded by a river that looks larger than lake ontario. What are the dimensions of your farm which give you the largest area to farm in?

(1) This problem is ridic

(2) Draw rectangle, with river

(3) Let s and t be the two sides, A the area and P the perimiter.

(4) Then

$$A = s \cdot t \quad P = 2s + t$$

We know that $P = 400$

(5) So $t = 400 - 2s$ and $A = s \cdot (400 - 2s)$.

(6) Recall from second derivative test, we want concave down and first derivative to be 0.

$$A' = 400 - 4s \quad A'' = -4$$

Now

$$A' = 0 \Rightarrow s = 100$$

So we have critical points at $s = 0$, $s = 100$, $s = 200$. Then

$$A(0) = 0 \quad A(200) = 0 \quad A(100) = 100(200) = 20,000$$

Therefore, when $s = 100$ we have maximal area, so the dimensions are:

$$(s =)100 \times 200(= t)$$

Exercise 10.17 You and a partner are in charge of Coca Cola Canada (C^3) and your bosses have said they are spending to much money on metal and want to come up with a smaller size. Keeping the can a cylindrical shape, what are the dimensions of the can that minimize surface area if you have 250ml of coke in each bottle.

Some help:

- Surface area can be calculated by calculating the area of the two circles (on top and on the bottom) and then adding the surface area of the cylinder part.
- The volume of the can is the same as the area of the top times the height.
- The volume of the can is 250.

(1) Let's try

(2) Draw a cylinder with radius r and height h.

(3) r, t, $A = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$.

(4) $h = \frac{250}{\pi r^2}$ and
$$A = 2\pi r^2 + 2\pi r\frac{250}{\pi r^2} = 2\pi r^2 + \frac{125}{r}$$

(5) Then
$$A' = 4\pi r - \frac{125}{r^2} = \frac{4\pi r^3 - 125}{r^2}$$
Then
$$A' = 0 \Rightarrow 4\pi r^3 = 125$$
so $r = \sqrt[3]{\frac{125}{4\pi}}$. Now
$$A'' = 4\pi + \frac{250}{r^3} > 0$$
which implies this is always concave up. Therefore, we have a local minmum at $r = \sqrt[3]{\frac{125}{4\pi}}$ and this is our only local minimum giving us our dimensions.

Week 11

16–20 Mar 2020

1 Day 1:16 Mar 2020

11.1 Recall:

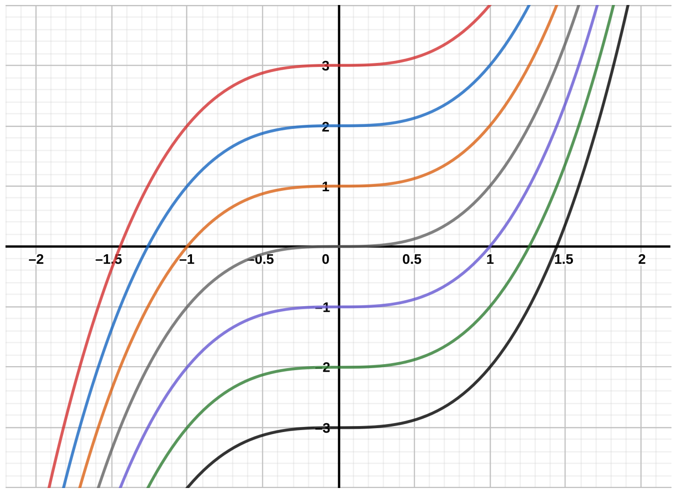
11.2 Antiderivatives - §4.9

We've been looking at how to find the derivative of a function $f(x)$. But what happens if we want to go backwards? Say we have some function $f(x)$ and we want to know whose derviative it is. This is the idea of antiderivatives.

Definition 11.1 A function F is called an *antideriative* of f on an interval I if $\underline{F'(x) = f(x)}$ for all x in I.

Example 11.2 If $f(x) = 3x^2$, what is its antiderivative? $\underline{x^3 + c}$

What do these look like?



But how do we know that's all? What if there is some other anti-derivative? This turns out not to be the case because of the following theorem.

Theorem 11.3 *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + c$$

where c is an arbitrary constant.

Example 11.4 What are the (most general) antiderivatives of the following functions:

(1) $f(x) = \cos(x)$: $\sin(x) + c$

(2) $f(x) = x^n$ where $n \neq 1$: $\frac{x^{n+1}}{n+1} + c$

(3) $f(x) = x^{-1}$: $\ln(|x|) + c$

We can use our derivative laws to help us find the antiderivatives of more complicated functions. For example, if we have $f(x) + g(x)$ and if $F(x)$ and $G(x)$ are the antiderivatives of f and g respectively, then $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$. Let's see an example.

Example 11.5 Find all antiderivatives of $f(x) = \frac{1}{\sqrt{1-x^2}} - 2\cos(x) + \frac{2x^5+\sqrt[3]{x}}{x^2}$

$$\begin{aligned}
\frac{1}{\sqrt{1-x^2}} &\to \arcsin(x) \\
-2\cos(x) &\to -2\sin(x) \\
\frac{2x^5+\sqrt[3]{x}}{x^2} &= \frac{2x^5}{x^2} + \frac{\sqrt[3]{x}}{x^2} \\
&= 2x^3 + x^{-5/3} \\
&\to \frac{2x^4}{4} + \frac{x^{-2/3}}{-2/3} \\
F(x) &= f'(x) \\
&= \arcsin(x) - 2\sin(x) + \frac{x^4}{2} - \frac{3x^{-2/3}}{2} + c \\
&= \arcsin(x) - 2\sin(x) + \frac{x^4 - 3x^{-2/3}}{2} + c
\end{aligned}$$

Exercise 11.6 Try one with a partner. Let $f(x) = 2\sin(x) - \frac{1+x^3}{x}$ and find all antiderivatives of $f(x)$.

$$F(x) = f'(x) = -2\cos(x) - \ln(|x|) + \frac{x^3}{3}$$

If we're looking for a particular answer, we also give a data point with our function.

Example 11.7 Find f if $f'(x) = e^x - 2\cos(x)$ and $f(0) = 2$.

$$f(x) = e^x - 2\sin(x) + c$$

But we know $f(0) = 2$. So

$$2 = f(0) = e^0 - 2\sin(0) + c = 1 - 0 + c = 1 + c$$

Thereofre, $c = 1$ and $f(x) = e^x - 2\sin(x) + 1$.

Let's try something a little harder.

Example 11.8 Find f if $f''(x) = 3x + 2$ where $f(0) = 4$ and $f(2) = 13$.

$$f'(x) = \frac{3x^2}{2} + 2x + c \qquad f(x) = \frac{x^3}{2} + x^2 + cx + d$$

Since $f(0) = 4$ then $4 = \frac{0^3}{2} + 0^2 + c \cdot 0 + d = d$.

Since $f(2) = 13$ then $13 = \frac{2^3}{2} + 2^2 + c \cdot 2 + 4 = 4 + 4 + 2c + 4 = 12 + 2c$ and $c = \frac{1}{2}$

$$f(x) = \frac{x^3}{2} + x^2 + \frac{x}{2} + 4$$

Exercise 11.9 With a partner, find f if $f''(x) = 4x^2 + 1$ where $f(0) = \frac{1}{6}$ and $f(1) = 2$.

$$f'(x) = \frac{4x^3}{3} + x + c \qquad f(x) = \frac{x^4}{3} + \frac{x^2}{2} + cx + d$$

$$f(0) = \frac{1}{6} = d \qquad 4(1) = \frac{1}{3} + \frac{1}{2} + c + \frac{1}{6} = c + 1$$

$$f(x) = \frac{x^4}{3} + \frac{x^2}{2} + x + \frac{1}{6}$$

2 Day 2: 18 Mar 2020

11.1 Recall:

Function	Derivative	Function	Derivative
$f(x) = x^r$	$f'(x) = rx^{r-1}$	$f(x) = kg(x)$	$f'(x) = kg'(x)$
$f(x) = g(x)h(x)$	$f'(x) = g'(x)h(x) + g(x)h'(x)$	$f(x) = \frac{g(x)}{h(x)}$	$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$
$f(x) = g(h(x))$	$f'(x) = g'(h(x))h'(x)$	$f(x) = g(x)^r$	$f'(x) = r(g(x))^{r-1}g'(x)$
$f(x) = e^x$	$f'(x) = e^x$	$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$
$f(x) = a^x$	$f'(x) = a^x \ln(a)$	$f(x) = \log_a(x)$	$f'(x) = \frac{1}{x\ln(a)}$
$f(x) = \sin(x)$	$f'(x) = \cos(x)$	$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
$f(x) = \tan(x)$	$f'(x) = \sec^2(x)$	$f(x) = \cot(x)$	$f'(x) = -\csc^2(x)$
$f(x) = \sec(x)$	$f'(x) = \sec(x)\tan(x)$	$f(x) = \csc(x)$	$f'(x) = -\csc(x)\cot(x)$
$f(x) = \arcsin(x)$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$	$f(x) = \arccos(x)$	$f'(x) = \frac{-1}{\sqrt{1-x^2}}$
$f(x) = \arctan(x)$	$f'(x) = \frac{1}{1+x^2}$	$f(x) = \text{arccot}(x)$	$f'(x) = \frac{-1}{\sqrt{1+x^2}}$
$f(x) = \text{arcsec}(x)$	$f'(x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$f(x) = \text{arccsc}(x)$	$f'(x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

3 Day 3: 20 Mar 2020

11.1 Recall:

Practice Midterm 2

Week 12

23–27 Mar 2020

1 Day 1: 23 Mar 2020

12.1 Recall:

Midterm 2

2 Day 2: 25 Mar 2020

12.1 Recall:

12.2 Summation - §Appendix E

We're going to kind of change topics. Remember that when we're looking at a graph, we used derivatives to find the slope at any point. We're now going to change our question. How can we calculate the area ?

Most of the time, it's easy. For a square we have the length times the height. For a triangle we have one half the base times the height. For a circle we have πr^2. But what if we want to calculate the area under a curve?

Before looking fully into this topic, we're going to take a quick detour into summations.

Example 12.1 Say that we want to add 1 to itself ten times. We can do it like: $1+1+1+1+1+1+1+1+1+1 = 10$. But this becomes more complicated if we want to do more than ten. Say we want to do it 25 times:

$$1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1 = 25.$$

This is way to long and almost imposible to read. So we want to rewrite this in a more visible manner. For adding 1 to itself, it's pretty easy: we do $1 \cdot n$ where n ist he number of times we want to add 1 to itself.

How about if we want to add every integer from 1 to 10: $1+2+3+4+5+6+7+8+9+10 = 55$. What if we want to do from 1 to 25?

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20+21+22+23+24+25 = \underline{325}$$

This is way to complicated and we want to condense this summation by introducing a new notation $\underline{\Sigma}$.

Example 12.2 Here are the two examples from before.

$$\sum_{i=1}^{10} 1 = 10 \qquad \sum_{i=1}^{10} i = 55$$

In general, we have something like the following

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

The i are called the *indices of summation* and they can be thought of as functions.

$$a_i = 1 \qquad a_i = i$$

We can also do this sum without end! If we want to take the summation forever then we let $n = \infty$: $\sum_{i=m}^{\infty} a_i$.

Example 12.3 Let's do some examples.

$$\sum_{i=m}^{n} a_i = \sum_{i=1}^{4} 2i - 1 = 2(1) - 1 + 2(2) - 1 + 2(3) - 1 + 2(4) - 1 = 2 + 4 + 6 + 8 - 4 = 16$$

$$\sum_{i=m}^{n} b_i = \sum_{i=3}^{5} i^2 + 1 = \underline{3^2 + 1 + 4^2 + 1 + 5^2 + 1 = 53}$$

$$\sum_{j=x}^{y} i_j = \sum_{j=-4}^{-1} 3 = \underline{3 + 3 + 3 + 3 = 12}$$

And in the other direction, we have

$$2 + 2 + 2 + 2 + 2 = \sum_{i=1}^{5} 2$$

$$5 + 8 + 11 + 14 = \underline{\sum_{i=1}^{4} 3i + 2} = \sum_{i=2}^{5} 3i - 1$$

$$1 + 0 - 1 - 2 - 3 = \underline{\sum_{i=-3}^{1} i}$$

Here rae some properties of summations using sigma notation:

Proposition 12.4 *(1)* $\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j$.

(2) $\sum_{i=m}^{n} a_i = \sum_{i=1}^{n-m+1} a_{i+m-1}$.

(3) $\sum_{i=m}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m-1} a_i$.

(4) *Pour* $c \in \mathbb{R}$, $\sum_{i=m}^{n} c = (n-m+1)c$.

(5) *Pour* $c \in \mathbb{R}$, $\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$.

(6) $\sum_{i=1}^{n} (a_i + b_i) = (\sum_{i=1}^{n} a_i) + (\sum_{i=1}^{n} b_i)$

(7) $\sum_{i=1}^{n} (a_i - a_{i-1}) = \underline{a_n - a_0}$. *telescoping sum*

(8) $\sum_{i=1}^{n} i = \underline{\frac{n(n+1)}{2}}$.

(9) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

(10) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

(11) $\sum_{i=1}^{n} r^i = \frac{r^{n+1}-r}{r-1}$.

Examples 12.5

$$\sum_{i=1}^{100} i = \frac{100 * (100+1)}{2} = \frac{10100}{2} = 5050$$

$$\sum_{k=5}^{12} k^2 = \underline{\frac{12(12+1)(2\cdot 12+1)}{6} - \frac{4(4+1)(2\cdot 4+1)}{6}} = 2\cdot 13\cdot 25 - \frac{4\cdot 5\cdot 9}{6} = 650 - 30 = 620$$

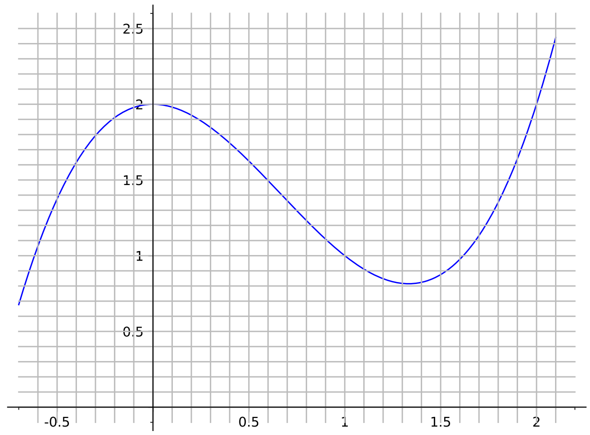
Exercise 12.6 Try one with a partner.

$$\sum_{i=1}^{20} i^2 - i - 10$$

$$\frac{20(21)(41)}{6} - \frac{20(21)}{2} - 10 * 20 = 2870 - 210 - 200 = 2460$$

12.3 Areas and distances - §5.1

Suppose we have the following function: $f(x) = x^3 - 2x^2 + 2$:



Since we already kinda know how to look at tangents to a curve, let's look at area under a curve. Say we want to calculate the area under the curve between 0 and 2. How can we do this?

$$\begin{aligned}&\sum_{i=0}^{3} f\left(\frac{i}{2}\right)\Delta x\\ &\quad= \left(2 + \frac{13}{8} + 1 + \frac{7}{8}\right)\frac{1}{2}\\ &\quad= \frac{11}{4} \approx 2.75000000000000\end{aligned}$$

$$\begin{aligned}&\sum_{i=0}^{9} f\left(\frac{i}{5}\right)\Delta x\\ &\quad= \left(2 + \frac{241}{125} + \frac{218}{125} + \frac{187}{125} + \frac{154}{125} + 1 + \frac{106}{125} + \frac{103}{125} + \frac{122}{125} + \frac{169}{125}\right)\frac{1}{5}\\ &\quad= \frac{67}{25} \approx 2.68000000000000\end{aligned}$$

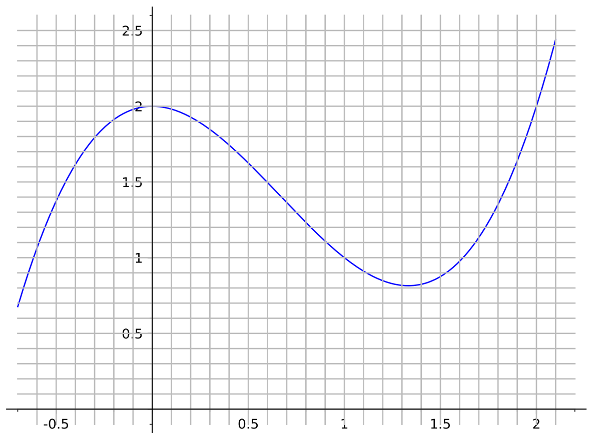
Using our techniques, we can estimate that the area should be: $\frac{8}{3} \approx 2.66666666666667$.

In essence, we are generally using the following formula for trying to calculate the area:

$$\sum_{i=0}^{n} f(x_i)\Delta x$$

But we can look at this function differently. We can get an estimation by looking at the right-hand sides of our rectangles:

$$\sum_{i=0}^{n} f(x_{i+1})\Delta x$$



Doing this with our example, we'd get:

$$\begin{aligned}
&\sum_{i=0}^{9} f\left(\frac{i+1}{5}\right)\Delta x \\
&\quad = \left(\frac{241}{125} + \frac{218}{125} + \frac{187}{125} + \frac{154}{125} + 1 + \frac{106}{125} + \frac{103}{125} + \frac{122}{125} + \frac{169}{125} + 2\right)\frac{1}{5} \\
&\quad = \frac{67}{25} \approx 2.68000000000000
\end{aligned}$$

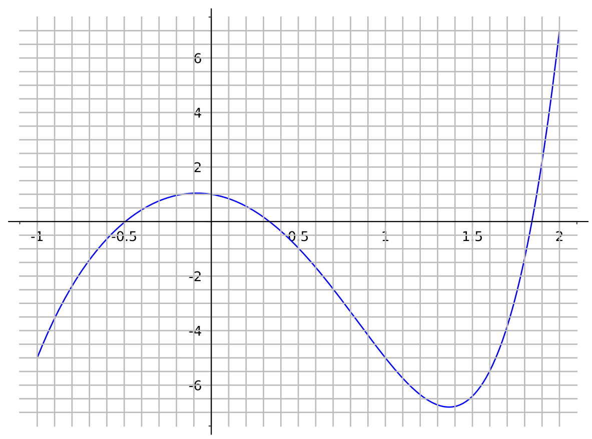
Notice how these two values are similar to one another!

3 Day 3: 27 Mar 2020

12.1 Recall:

12.2 Definite intgral - §5.1 – 5.2

Now let's look at a different function, and we're going to be a little more precise in how we are defining these rectangles. We will work with $f(x) = x^5 - 6\,x^2 - x + 1$:



Given a closed interval $[a, b]$, a *partition* of an interval of numbers x_i is such that:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

This gives us little sub-intervals:

$$[x_0, x_1],\ [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

If the intervals are all the same size $(\frac{b-a}{n})$ we say that the partition is *regular*.

Example 12.7 For some examples, let $a = 2$ and $b = 5$. The following are two different partitions:

$$2 < 2.1 < 3 < 4.2 < 4.5 < 4.51 < 5$$
$$2 < 3 < 4 < 5$$

and they each define different intervals:

$$[2, 2.1],\ [2.1, 3],\ [3, 4.2], [4.2, 4.5],\ [4.5, 4.51],\ [4.51, 5]$$
$$[2, 3], [3, 4],\ [4, 5]$$

Which partition is regular? 2

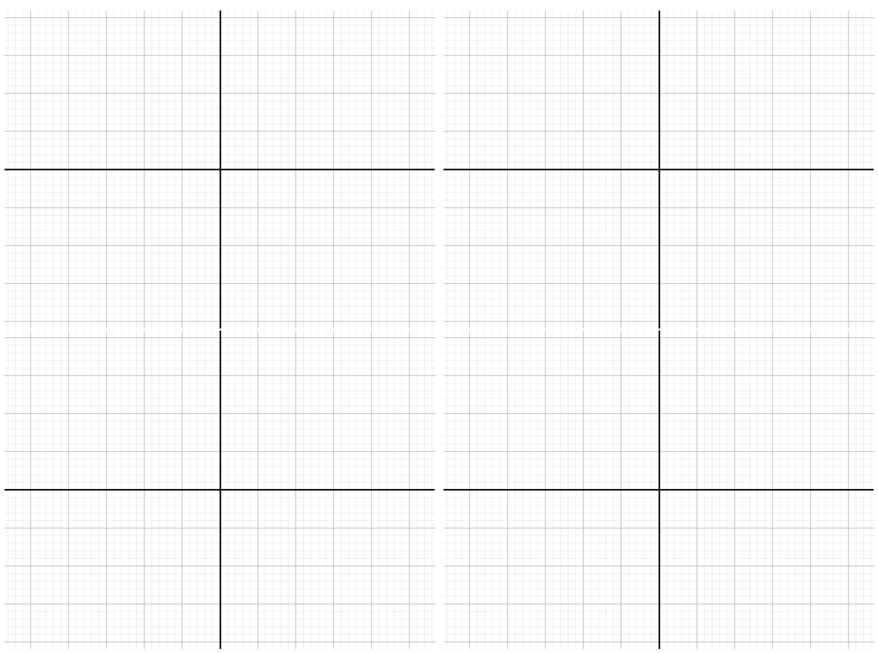
Exercise 12.8 Given $a = 0$ and $b = 1$, with a partner, give a regular partition: $0 < 1$

Always assume regular The *length of an interval* is the difference between the start and end of the interval: $\Delta x_i = \underline{x_i - x_{i-1}}$ for the interval $[x_{i-1}, x_i]$.

We are now ready to define our summations! The *Riemann sum* of a function on an interval $[a, b]$ is the sum:

$$\sum_{i=1}^{n} f(x_i^\star)\Delta x_i$$

where $x_i^\star \in [x_{i-1}, x_i]$ is some arbitrary number.



Show left/right/mid and function

We've already seen two of these! The *right-hand sum* is the Riemann sum where we let $x_i^\star = x_i$. The *left-hand sum* is the Riemann sum where we let $x_i^\star = x_{i-1}$. The *midpoint sum* is the Riemann sum where we let $x_i^\star = \frac{x_{i-1}+x_i}{2}$.

But what we reeeeaaaallllly want is to find the *exact* area, not just approximations! What can we do to find the exact area?

We're going to use our favorite tool so far in class: limits. Yes... limits.

The limit we're going to use is actually the following limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x_i$$

If this limit exists we call say that the function is *Riemann-integrable*.

BUT, this notation is horrible. There are a million and a half symbols. Who wants to write that each time? No one. So we're going to use another notation that (I think) everyone has already seen: the integral.

Remark 12.9 Some history! We use Σ for summation because summation starts with s. So, we need a new symbol for the integral. Luckily, Gottfried Leibniz (German) had came up with the perfect notation back in 1675! He used ancient German's long s: ſ. (Nowadays, German's long s is written as "ß".) Why? Because the integral is a summation of realy small intervals. So ſ is perfect for that since it keeps it as an "s", but is a different s! (Even in English we used to have ſ! It wasn't until between 1800 and 1820 that this letter disappeared from English.)

We changed the "s", but we also need to change the Δ since we're taking the limit. Since we're taking the limit, these intervals are getting smaller and smaller. In other words, it would be nice to represent that by using a "small" Δ. So what's the lower-case of Δ? It's δ of course! But, Leibniz, who was one of the founders of calculus, translated the greek to German and the δ became a d. This is where Leibniz notation: $\frac{dy}{dx}$ actually came from!

So, *if* our function is Riemann-integrable, we will write:

$$\underline{\int_a^b f(x)\,dx} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x_i.$$

Some definitions:

- The a is the *lower limit of integration*,

- The b is the *upper limit of integration*,
- The $f(x)$ is the *integrand*,
- The x is called the *variable of integration* or the *independent variable*.

Theorem 12.10 *If a function $f(x)$ is continuous on the interval $[a, b]$ or if $f(x)$ has only a finite number of jump discontinuities, then the function $f(x)$ is (Riemann)-integrable on $[a, b]$.*

Example 12.11 Let's look at an (easy) example. Find $\int_1^2 x^2\,dx$. By definition

$$\int_1^2 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^\star)\Delta x_i$$

What is Δx_i? Length of (regular) interval.

$$\Delta x_i = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

What is $x_i^\star$? We can chose, but let's use right-hand:

$$x_i^\star = x_i = 1 + \frac{i}{n} = \frac{n+i}{n}$$

What is $f(x_i^\star)$?

$$f(x_i^\star) = f(\frac{n+i}{n}) = \left(\frac{n+i}{n}\right)^2 = \frac{n^2+2ni+i^2}{n^2}$$

So

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n^2+2ni+i^2}{n^2}\cdot\frac{1}{n}$$
$$\begin{aligned}
&= \lim_{n\to\infty}\frac{n^2}{n^3}\sum_{i=1}^{n}1+\frac{2n}{n^3}\sum_{i=1}^{n}i+\frac{1}{n^3}\sum_{i=1}^{n}i^2\\
&= \lim_{n\to\infty}\frac{n}{n}+\frac{2}{n^2}\frac{n(n+1)}{2}+\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\\
&= \lim_{n\to\infty}1+\frac{2(n^2+n)}{2n^2}+\frac{2n^3+3n^2+n}{6n^3}\\
&= \lim_{n\to\infty}1+1+\frac{1}{n}+\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\\
&= 1+1+0+\frac{1}{3}+0+0\\
&= \frac{7}{3}
\end{aligned}$$

That was disgusting. The whole point of the next few days (and the next class) is to try and make this easier.

12.3 Properties of definite integrals - §5.2

Proposition 12.12 *Let $f(x)$ be a continuous function on $[a,b]$ and let c be a constant.*

(1) $\int_a^b f(x)\,dx = \int_a^b f(t)dt = \int_a^b f(u)du$

(2) $\int_a^a f(x)\,dx = \underline{0}$

(3) $\int_a^b c\,dx = c(b-a)$

(4) $\int_a^b f(x)\,dx = \underline{-\int_b^a f(x)\,dx}$

(5) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(6) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(7) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \underline{\int_a^c f(x)\,dx}$

Week 13

30 Mar – 3 Apr 2020

1 Day 1: 30 Mar 2020

13.1 Recall:

13.2 The fundamental theorem of calculus - §5.3

Remember that in summations we had one summation which we called the telescoping sum:

$$\sum_{i=1}^{n} a_i - a_{i-1} = \underline{a_n - a_0}.$$

In fact, our integral is written in almost the same way:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)(x_i - x_{i-1}).$$

So we now want to know if there exists an easier way to solve these integrals and we will see (very soon) that a way exists! In fact, this is done through antiderivatives!

Recall that an *antiderivative* of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Theorem 13.1 (The fundamental theorem of calculus) *If $f(x)$ is a continuous function on an interval $[a, b]$ and if $F(x)$ is an antiderivative of $f(x)$ then*

$$\int_a^b f(x)\,dx = \underline{F(b) - F(a)} = F(x)\,\Big|_a^b$$

To make life easier, here are some functions with some antiderivatives

Function	**(an) antiderivative**
x^n where $n \neq -1$	$\frac{x^{n+1}}{n+1}$
x^{-1}	$\ln(\lvert x \rvert)$
$\sin(kx)$ where $k \neq 0$	$-\frac{\cos(kx)}{k}$
$\cos(kx)$ where $k \neq 0$	$\frac{\sin(kx)}{k}$
$\sec^2(kx)$ where $k \neq 0$	$\frac{\tan(kx)}{k}$
$\operatorname{cosec}^2(kx)$ where $k \neq 0$	$-\frac{\operatorname{cotan}(kx)}{k}$
$\sec(kx)\tan(kx)$ where $k \neq 0$	$\frac{\sec(kx)}{k}$
$\operatorname{cosec}(kx)\operatorname{cotan}(kx)$ where $k \neq 0$	$-\frac{\operatorname{cosec}(kx)}{k}$
e^{kx} where $k \neq 0$	$\frac{e^{kx}}{k}$

Example 13.2 Let's look at some examples:

$$\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8-1}{3} = \frac{7}{3}$$

$$\int_2^4 2x\,dx = \underline{2 \cdot \left.\frac{x^2}{2}\right|_2^4 = 2 \cdot \left(\frac{4^2}{2} - \frac{2^2}{2}\right) = 2 \cdot \left(\frac{16-4}{2}\right) = 16 - 4 = 12}$$

$$\int_0^\pi \sin(x)\,dx = \underline{-\cos(x)|_0^\pi} = -\cos(\pi) + \cos(0) = 1 + 1 = 2$$

$$\int_5^5 e^{x^2} \cdot \sin^2(x^2)\,dx = \underline{0}\text{We don't know how to do this!}$$

Exercise 13.3 Try an example with a partner. Solve:

$$\int_1^3 4x^3 - e^x + \frac{1}{x}\,dx$$

Hint: Recall the properties of integrals from before.

$$\begin{aligned}\int_1^3 4x^3 - e^x + \frac{1}{x}\,dx &= \int_1^3 4x^3\,dx - \int_1^3 e^x\,dx + \int_1^3 \frac{1}{x}\,dx \\ &= \frac{4x^4}{4}\bigg]_1^3 - e^x\big]_1^3 + \ln(|x|)\big]_1^3 \\ &= 3^4 - 1^4 - e^3 + e^1 + \ln(3) - \ln(1) \\ &= 80 - e^3 + e + \ln(3)\end{aligned}$$

13.3 Indefinite integrals - §5.4

Because of the fundamental theorem of calculus, we can find the integral $\int_a^b f(x)\,dx$. But in order to find this, we must first find an antiderivative of $f(x)$!

This is not always easy!

Example 13.4 Find a function $F(x)$ such that $F'(x) = f(x) = e^{x^2}$. Way to difficult for us.

For this, we need to find a way to generate antiderivatives. This is normally done by using indefinite integrals.

A *indefinite integral* is the set of all antiderivatives of a given function. In other words, it's the general formula $F(x) + c$ that we saw a few days ago. We will normally denote this by:

$$\int f(x)\,dx = F(x) + c$$

All this is saying is that $F(x) + c$ is the general formula for an antiderivative of $f(x)$.

Note: Definite and indefinite integrals are different!!! Definite integrals have lower/upper limits and give you a value. Indefinite integrals have no lower/upper limits and give you a function.

We already have a ton of indefinite integrals that we can create using the derivative rules we saw earlier this semester:

(1) $\int k\,dx = kx + c$

(2) $\int [k_1 f(x) + k_2 g(x)]\,dx = k_1 \int f(x)\,dx + k_2 \int g(x)\,dx$

(3) $\int x^n\,dx = \underline{\frac{x^{n+1}}{n+1} + c}$ (where $n \neq -1$)

(4) $\int \frac{1}{x}\,dx = \ln(|x|) + c$

(5) $\int e^x\,dx = \underline{e^x + c}$

(6) $\int a^x\,dx = \frac{a^x}{\ln a} + c$ (where $1 \neq a > 0$)

(7) $\int \sin(x)\,dx = \underline{-\cos(x) + c}$

(8) $\int \cos(x)\,dx = \underline{\sin(x) + c}$

(9) $\int \sec(x)\,dx = \ln(|\sec(x) + \tan(x)|) + c$

(10) $\int \operatorname{cosec}(x)\,dx = \ln(|\operatorname{cosec}(x) - \operatorname{cotan}(x)|) + c$

(11) $\int \tan(x)\,dx = \ln(|\sec(x)|) + c$

(12) $\int \operatorname{cotan}(x)\,dx = \ln(|\sin(x)|) + c$

(13) $\int \sec^2(x)\,dx = \tan(x) + c$

(14) $\int \operatorname{cosec}^2(x)\,dx = -\operatorname{cotan}(x) + c$

(15) $\int \sec(x)\tan(x)\,dx = \sec(x) + c$

(16) $\int \operatorname{cosec}(x)\operatorname{cotan}(x)\,dx = -\operatorname{cosec}(x) + c$

(17) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c$

(18) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(|x|) + c$

(19) $\int \frac{1}{1+x^2}\,dx = \arctan(x) + c$

But who cares? We care because it gives us an easier way to find the definite integral when we want to find it.

Example 13.5 Evaluate the following expression.

$$\begin{aligned}
\int_2^4 \frac{12}{x^3}\,dx &= 12\int_2^4 \frac{1}{x^3}\,dx \\
&= 12 \cdot \left.\frac{1}{-2x^2}\right|_2^4 \\
&= 12 \cdot \left(\frac{1}{-2\cdot 4^2} - \frac{1}{-2\cdot 2^2}\right) \\
&= 12 \cdot \left(\frac{1}{-32} - \frac{1}{-8}\right) \\
&= 12 \cdot \left(\frac{4}{32} - \frac{1}{32}\right) \\
&= 12 \cdot \frac{3}{32} \\
&= \frac{9}{8}.
\end{aligned}$$

Exercise 13.6 Try and find the following limit with a partner.

$$\int_1^4 \frac{1}{\sqrt{x^3}}\,dx$$

$$\begin{aligned}
&= \int_1^4 \frac{1}{x^{3/2}}\,dx \\
&= \left.\frac{1}{(-1/2)x^{1/2}}\right|_1^4 \\
&= \left.\frac{-2}{\sqrt{x}}\right|_1^4 \\
&= \left(\frac{-2}{\sqrt{4}} - \frac{-2}{\sqrt{1}}\right) \\
&= \frac{-2}{2} - (-2) \\
&= 2 - 1 \\
&= 1
\end{aligned}$$

Example 13.7 Let's try a more complicated example. Let's find $\int_0^\pi 3x^2 - e^x + \cos(x)\,dx$.

$$\int 3x^2 - e^x + \cos(x)\,dx = \underline{3\int x^2\,dx - \int e^x\,dx + \int \cos(x)\,dx} = \frac{3x^3}{3} - e^x + \sin(x)$$

Therefore

$$\begin{aligned}
\textstyle\int_0^\pi 3x^2 - e^x + \cos(x)\,dx \quad &= \left.\big(x^3 - e^x + \sin(x)\big)\right|_0^\pi \\
&= \big(\pi^3 - e^\pi + \sin(\pi)\big) - \big(0^3 - e^0 + \sin(0)\big) \\
&= \pi^3 - e^\pi + 0 - 0 + 1 - 0 \\
&= \pi^3 - e^\pi + 1
\end{aligned}$$

2 Day 2: 1 Apr 2020

13.1 Recall:

13.2 Substituion - §5.5

We're now going to look at the chain rule, but for integration. So suppose we have some (indefinite) integral like: $\int (3x+2)^{7/9}\,dx$? And we're trying to find the antiderivative of $(3x+2)^{7/9}$.

So far we haven't really learned how to tackle something like this, but we can use the ideas of the chain rule (and implicit differentiation) to solve something like this. What this method is called is "substitution".

What we want is an integral that is easier to handle. In our case, we want to change our integral to be something like $\int u^{7/9}\,du$. Why? Because we already know how to do this function!

So if we set $u = 3x+2$, then we have $\int u^{7/9}\,dx$. But now we have a different problem. What do you think the problem is? Variables don't match up. (have them try it)

Well, let's see if we can figure it out! We're gonna use implicit differentiation:

$$u = 3x+2 \Rightarrow \underline{du = 3\,dx} \Rightarrow \; dx = \frac{du}{3}$$

Therefore

$$\begin{aligned}
\int (3x+2)^{7/9}\,dx &= \underline{\int u^{7/9}\frac{du}{3}} \\
&= \frac{1}{3}\int u^{7/9}\,du \\
&= \underline{\frac{1}{3}\frac{u^{16/9}}{16/9}+c} \\
&= \frac{9}{3\cdot 16}u^{16/9}+c \\
&= \frac{3}{16}(3x+2)^{16/9}+c
\end{aligned}$$

And done!

So what did we do?

(1) Set $u = f(x)$

(2) Do implicit differentiation

(3) Replace the variables

(4) Do the integrals

(5) Re-replace the variables

Exercise 13.8 With a partner, find the following integral: $\int x(7x^2-8)^{16/7}\,dx$. **Hint:** Don't worry about the x by itself.

$$u = 7x^2 - 8 \Rightarrow du = 14x\,dx \Rightarrow\ dx = \frac{du}{14x}$$

Then

$$\begin{aligned}
\int x(7x^2-8)^{16/7}\,dx &= \int x(u)^{16/7}\,dx \\
&= \int x(u)^{16/7}\frac{du}{14x} \\
&= \frac{1}{14}\int u^{16/7}du \\
&= \frac{1}{14}\frac{u^{23/7}}{23/7} + c \\
&= \frac{7}{14*23}u^{23/7} + c \\
&= \frac{1}{46}u^{23/7} + c \\
&= \frac{1}{46}\left(7x^2-8\right)^{\frac{23}{7}} + c
\end{aligned}$$

But recall that we're doing all of these things to find definite integrals! So what happens with that?

Example 13.9 Find the following definite integral: $\int_2^4 \frac{2x-1}{x^2-x}\,dx$

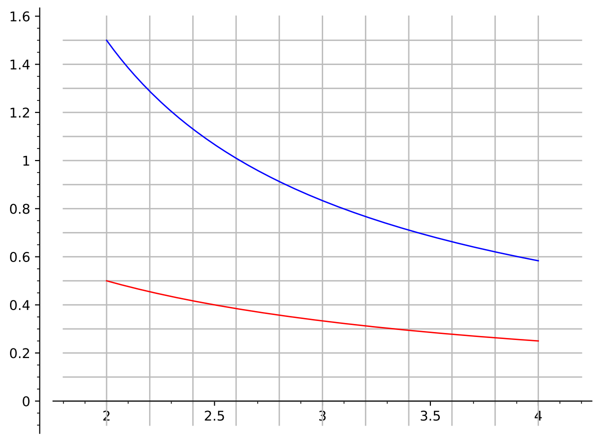
$$u = x^2 - x \Rightarrow du = (2x-1)\,dx \Rightarrow\ dx = \frac{du}{2x-1}$$

Therefore

$$\int_2^4 \frac{2x-1}{x^2-x}\,dx = \int_2^4 \frac{2x-1}{u}\,\frac{du}{2x-1} = \int_2^4 \frac{1}{u}\,du$$

But nowe we have a different problem!
The limits are not correct!

Let's look at the two functions together. The one on top (blue) is the one with variable x and the one on the bottom (red) is the one with variable u.



So how do we solve this? We do exactly like we did with dx. If $x = 4$ then $u = x^2 - x = 4^2 - 4 = 16 - 4 = 12$.

If $x = 2$ then $u = 2^2 - 2 = 4 - 2 = 2$.

Then

$$\begin{aligned}\int_2^4 \frac{2x-1}{x^2-x}\,dx &= \int_2^{12} \frac{1}{u}\,du\\ &= \ln(|u|)\big|_2^{12}\\ &= \ln(12) - \ln(2)\\ &= \ln(12/2) = \ln(6)\end{aligned}$$

Here's the substitution rule in all its glory:

Theorem 13.10 (Substitution rule for a definite integral) *If $g'(x)$ is a continuous function on the interval $[a, b]$ and if $f(x)$ is a continuous function that has an antiderivative on the interval $[g(a), g(b)]$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \underline{\int_{g(a)}^{g(b)} f(u)du}$$

Where does the $g'(x)$ come from? From the chain rule!

$$\begin{aligned} u = g(x) \Rightarrow\ du &= g'(x)\, dx \\ &\Rightarrow \frac{du}{g'(x)}\, dx \end{aligned}$$

and therefore to be able to change $f(g(x))$ to $f(u)$ we must always have $g'(x)$ there.

Example 13.11 Let's try this with a function that's a little bit more fun $\int \text{cotan}(ax)\, dx$ for $a \in \mathbb{R}$.

$$u = ax \Rightarrow\ dx = \frac{du}{a}$$

Then

$$\int \text{cotan}(ax)\, dx = \frac{1}{a} \int \text{cotan}(u)\, du = \frac{1}{a} \int \frac{\cos(u)}{\sin(u)}\, du$$

Recall that $\frac{d}{dx}(\sin(x)) = \cos(x)$ Then

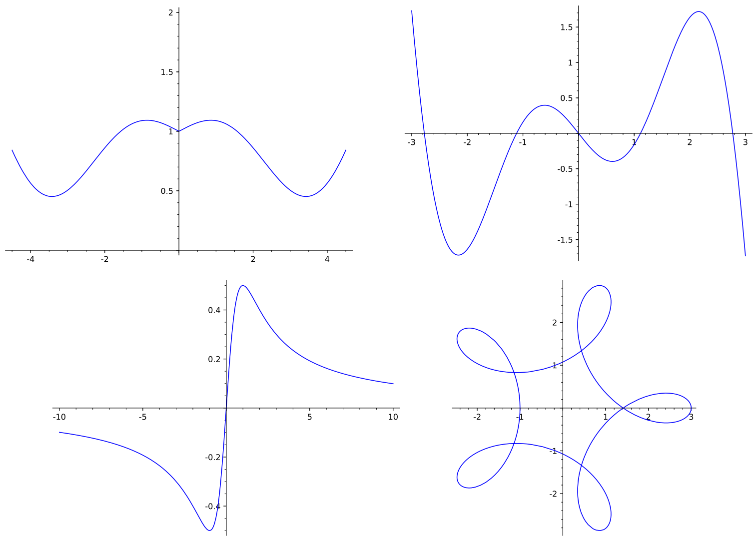
$$t = \sin(u) \Rightarrow du = \frac{dt}{\cos(u)}$$

and

$$\begin{aligned} \frac{1}{a} \int \frac{\cos(u)}{\sin(u)}\, du &= \frac{1}{a} \int \frac{\cos(u)}{t} \frac{dt}{\cos(u)} \\ &= \frac{1}{a} \int \frac{1}{t}\, dt \\ &= \frac{1}{a} \ln(|t|) + c \\ &= \frac{1}{a} \ln(|\sin(u)|) + c \\ &= \frac{1}{a} \ln(|\sin(ax)|) + c \end{aligned}$$

13.3 Even and odd functions - §5.5

Recall that a function is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. Some examples:



Theorem 13.12 (Properties of even and odd functions) *If $f_1(x)$ and $f_2(x)$ are even functions and if $g_1(x)$ and $g_2(x)$ are odd functinos, then, where the following operations are defined, we have:*

(1) $f_1(x)f_2(x)$ *and* $g_1(x)g_2(x)$ *are* even *functions.*

(2) $f_1(x)g_1(x)$ *is an* odd *function.*

(3) $\int_{-a}^{a} f_1(x)\,dx = 2\int_0^a f_1(x)\,dx$

(4) $\int_{-a}^{a} g_1(x)\,dx = 0$

Example 13.13 For example:

$$\int_{-a}^{a} x^2\sin(x) - x^7 + x\cos(x)\,dx = 0$$

3 Day 3: 3 Apr 2020

13.1 Recall:

Practice Final

Week 14

Final Exam

The final exam will be Mon 13 April at 14h00 in $\cdots$