Mathematics I

Class Notes - Student Version

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Welcome

Welcome to Mathematics I! This semester we'll be going through a ton of various stuff in order to get you better equipped with the tools of linear algebra and analysis. These notes are designed to better help you learn the material and guide you through the course.

How to use these notes: These notes are different from most math class notes. They have holes and gaps in them; they are missing information! These are the notes that I will be using in my class lectures (literally) and so it should help you follow along. The purpose of these notes is so that you don't have to write everything down; you only need to fill in the gaps giving you more time to listen to lectures and better understand the material. If you need to add extra notes, there are tons of room in the margins, use them! These are YOUR notes. I've also added links to Wikipedia in case you want another reference to the terms and definitions we go over. If you feel using these notes would hinder your learning experience, then please use another system. The point is to learn; so use whatever works best for you.

Before starting a lecture I'd recommend quickly glancing over the notes (5-10 minutes) in order to get a brief idea of what we'll be covering in class. Remember to review these notes periodically so that you don't forget terms and examples.

Good luck, I believe in you!

-Aram

Chapter 1

Complex numbers

1 Numbers

We work with numbers every day of our lives. Whether we're doing our finances to make sure our bank account doesn't go empty, calculating the probability that we'll get a good grade in a class, or figuring out how much of our lives we've been doomscrolling on social media. Numbers have a rich history and there are many different types and sets of numbers exist.

Usually when we think of numbers, we're thinking of what are known as the natural numbers. The *natural numbers* are just the normal numbers we use for counting.

Wikipedia: natural numbers

$$1, 2, 3, 4, 5, \ldots$$

We normally use $\mathbb{N}$ to denote all of the natural numbers.

Depending on the book you look at, some books will state that 0 is also a natural numbers. This is a massive debate in the mathematical community, but for us, we will generally assume 0 is not a natural numbers.

Although the number 0 is now very familiar to many of us, it wasn't until the Indian mathematician Brahmagupta introduced it in 628 that it was used in mathematics. This number was even mysterious to Brahmagupta who made the classic error in his book, writing that __________. Incidentally, negative numbers were used much earlier and appear as numbers many centuries before the number 0.

The "idea" of zero was around many centuries before Brahmagupta with many civilizations either using a dot to represent zero or leaving a blank space. But in all these cases, zero was not treated as a number, but instead just as a concept.

If we add 0 to the natural numbers and also include the negative versions of all the natural numbers, we get what are known as the *integers* or as the *whole numbers*. We denote this set by $\mathbb{Z}$ and contains

Wikipedia: integers

the following elements as an example.

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

We use the letter $\mathbb{Z}$ because in German the word for "numbers" is "Zahlen".

Fractions are also very old with Babylonians having a system of fractions as early as 1800 BC. The horizontal bar commonly used for fractions was first used in the work by the Moroccan mathematician Abu Bakr al-Hassar in 1200 (before that, numbers just appeared one on top of the other without any bars). A fraction is nothing more than one whole number divided by another whole number: $\frac{p}{q}$ where $p, q \in \mathbb{Z}$. This gives us another set of numbers which we call the *rational numbers* and we denote them as $\mathbb{Q}$. Here are some examples of rational numbers:

Wikipedia: rational numbers

Here $\mathbb{Q}$ comes from the french "quotient" and was introduced by Bourbaki

$$\frac{1}{2}, -2, \frac{22}{7}, \frac{-13}{37}, 0, \frac{-1}{99}, \ldots$$

In around 500 BC Greek mathematicians realized that some numbers, like $\sqrt{2}$ were not rational. In other words, they couldn't be written as a fraction of two whole numbers. Instead of viewing them as numbers, they were ignored and not considered as numbers. The acceptance of fractions as numbers came about in the middle ages thanks to Indian and Chinese mathematicians. These numbers were considered as "numbers of measurement". In other words, if you have a ruler, you can measure that amount. Any number that can be measured was called a *real number* . These included all fractions, but also included other lengths which hadn't appeared so far. We denote the set of real numbers using $\mathbb{R}$ and they include all the rationals and other numbers such as:

Wikipedia: real number

$$\pi, \sqrt{3}, \varphi, e, -\sqrt{7}, \ldots$$

Numbers which are real, but are not rational (like the list above) are commonly referred to as *irrational* numbers.

Wikipedia: irrational

But what about the complex numbers? Let's first talk about their history.

2 History

Complex numbers began to appear in mathematics in the 1st century AD when the Greek mathematician Hero of Alexandria was trying to calculate the volume of frustums of a pyramid. Doing his normal calculations, he noticed something very strange happen. He arrived at the term $\sqrt{81 - 144} = \sqrt{-63}$. To him, this made absolutely no sense. He asked himself, "How can a number, when multiplied by itself, give a negative number?! The multiplication of two negative numbers is always positive." Thinking this was impossible, he simply just deleted the $-$ sign from his work and used $\sqrt{63}$ instead.

Although we do it frequently accidentally, removing a $-$ for absolutely no reason is not a good idea.

The topic wasn't really studied anymore until Italian mathematicians came to the scene in the 16th century. Mathematics at the time was crazy about finding roots of cubic and quartic polynomials. These are polynomials which look like $x^3 + 2x^2 + x + 3$ and $x^4 - 7x^2 + 2x - 14$. They would send each other puzzles to try and prove that they were "better" than the other. So the big race was to try and find a general formula like already existed in the quadratic case.

In the quadratic case, the general formula for finding roots of a polynomial $ax^2 + bx + c$ is given by $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which you've probably seen before.

The mathematician Scipione del Ferro was able to make a general formula for cubic functions, but in 1526 he died, taking his secrets on how to solve them to his grave. Then in 1535, the Venitian Niccolò Fontana Tartaglia rediscovered the methods of del Ferro and revealed his ideas to the Milanese Girolamo Cardano (who swore he would keep them a secret). Although he made this promise, Cardano broke his promise when he published Tartaglia's results in *Ars Magna* in 1545. The book gave the rules in a geometric way, acknowledging that Tartaglia had communicated the rules to him and that Tartaglia had learned the rules from del Ferro. Tartaglia was furious that Cardano broke his promise, causing an all out (math) war between the two. Cardano would end up winning this war as we now call this method of solving cubic equations the "Cardan method".

The equations that del Ferro discovered (and Tartaglia rediscovered and Cardano stole) had something hidden inside of them that would cause the same problem Hero of Alexandria had. The issue was that, even when your roots are all real and distinct, most of the time you would have negative numbers inside of the square root! It was ridiculous. Even Cardano didn't like working with them calling these numbers "as subtle as they are useless" and "mental torture". It wasn't until the Bolognese mathematician Raffael Bombelli that negative numbers in square roots began to be considered and studied.

In his book "L'algebra" from 1572, Bombelli was the first to systematically study negative numbers in square roots. His book was so popular that even Leibniz (who was one of the founders of Calculus) and Euler (who has a million theorems named after him) read and quoted Bombelli's book. He disliked it so much that he called them "più radice di meno" which means "plus the root of minus". This was abbreviated down to *p.dm.* and so $\sqrt{-2}$ would be denoted as *p.dm.*2. Complex numbers were finally making their way into mathematics, but they still had a long way to go.

The first thing that was needed was that the $\sqrt{-1}$ needed a name. The term "imaginary" for $\sqrt{-1}$ came around in 1637 thanks to René Descartes. It then needed a symbol, which was given by Euler in 1770 who used the letter i since he kept making errors when working with $\sqrt{-1}$. Although we now had a name and a symbol, nobody in the mathematical community really used either until 1867 when Gauss used it, solidifying their usage from then on.

3 Complex numbers

Now that we know the history of imaginary numbers, we can make use of the plethora of research that has been done to study these numbers in the last few centuries. To begin, let's define what a complex number even is.

Wikipedia: imaginary number

As mentioned we let $i = \sqrt{-1}$ be called an *imaginary number* . One of the best way to see a need for such a number is to try and work out what number solves the following equation:

$$x^2 + 1 = 0$$

Obviously $\pm\sqrt{-1}$ solves equation and so i is an important part of mathematics that helps us solve polynomial equations that we couldn't solve normally. This leads us to our first definition where we define everything formally.

Wikipedia: complex number

Definition 3.1 A *complex number* z is an ordered pair $z = (x, y)$ of real numbers $x, y \in \mathbb{R}$. We say that x is the *real part* of z and denote it as $\mathrm{Re}(z) = x$. Similarly, we say that y is the *imaginary part* of z and denote it as $\mathrm{Im}(z) = y$. We let $\mathbb{C}$ denote the set of complex numbers. Furthermore, we call $(0, 1)$ the *imaginary unit* and denote it by i.

Example 3.2 Consider $i = (0, 1)$. Well, we know that i is representing $\sqrt{-1}$ and so $i^2 = -1$. In the complex world, this would look like $i^2 =$ ____________. Therefore, $\mathrm{Re}(i^2) =$ ______ and $\mathrm{Im}(i^2) =$ __, which makes sense! We should have that i^2 only has a real part because it's a real number!

Example 3.3 Consider $\sqrt{-4}$. How do we write it as a complex number?

Well,

$$\sqrt{-4} =$$

Then $\mathrm{Re}(\sqrt{-4}) =$ __ and $\mathrm{Im}(\sqrt{-4}) =$ __.

This also allows us to solve for arbitrary quadratic equations!

Example 3.4 For example, let's find the roots of

$$x^2 + 8x + 25 = 0$$

By the quadratic formula we have:

$$x =$$

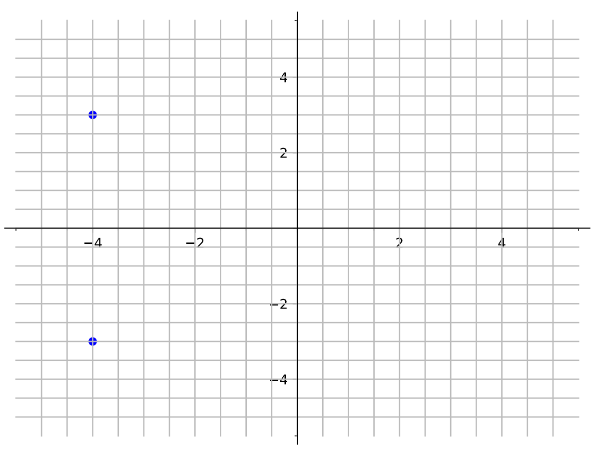
Converting this to our notation, we have that our two solutions are ____________ and _______________.

4 Representations

4.1 Graphical representation

Thinking of a complex number as a pair of numbers is a little bit weird, but if you think about it, fractions are the same thing. They are two numbers, but instead of putting them inside parenthesis, we put one on top of another. Writing complex numbers as a pair originated due to its geometric properties. How is this geometric? Well, it's a pair of numbers! And what do we do with a pair of numbers? We can plot it on a graph. For now, this doesn't make sense why we'd want to do that, but it'll be helpful for us in the future.

Example 4.1 Let's look at our example from before, where we had the roots of $x^2 + 8x + 25 = 0$ being $(-4, 3)$ and $(-4, -3)$. We can plot this to get these two points:



This gives us the *graphical representation* of a complex number. Note that in this case, the y-axis is referred to as the *imaginary axis* and the x-axis is called the *real axis*. To differentiate points on a graph and complex numbers we will use (x, y) to denote the complex number and $P(x, y)$ to denote the point. For a complex number (x, y), its associated point $P(x, y)$ is sometimes referred to as the *point associated to z*.

4.2 Binomial representation

Definition 4.2 Although we haven't given a name to it, there is a second way of talking about complex numbers. For a complex number $z = (x, y)$, the *binomial form* of z is given by $x + iy$. The binomial form is the standard way of working with complex numbers, and we will generally use that notation.

Given $z = x+iy$. If $y = 0$, then $z = x \in \mathbb{R}$ and so it's a real number. If $x = 0$, then $z = iy$ and it's called a *pure imaginary number*. Notice that using this representation helps us differentiate real and imaginary better than if we were to use coordinates.

4.3 Polar, trigonometric and exponential representations

Using the graphical representation, we can try and use polar coordinates in order to give us more representations of complex numbers. Since polar coordinates are weird, let's first set up the groundwork.

Recall first that for polar coordinates we need a length. The *modulus* $|z|$ of a complex number $z = x + iy$ is the distance from the origin in the graphical representation. But we can easily calculate this using our graph. We can see that the length is nothing more than ________________. In other words, we have

$$|z| =$$

Since it's a length, it will always be a real number.

Next, we need to figure out the angle it makes with the positive real axis. This angle is known as the *argument* of z and is denoted by $\arg(z)$. But we need to know how to calculate it.

Wikipedia: argument

Let $z = x+iy$ be a complex number. Since we know x and y, we know that __________________. Solving for θ gives $\theta =$ __________________.

But, here we need to be careful because the argument is the angle with the *positive* real axis. This means that, depending on the coordinate that we choose, we might need to add something to θ to get the appropriate angle.

In the first quadrant, it's not so bad, this θ coincides with the argument. In other words, we have.

$$\arg(z) = \theta = \qquad\qquad .$$

In other quadrants, since tan is periodic with period π, θ will be correct up to the adding of a multiple of π.

Example 4.3 Consider first $z = 1 + i$, calculate its modulus and argument.

For the modulus we have:

$$|z| =$$

For the argument we have:

$$\theta =$$

Now we must think about the point associated to z. This point is the point ____________ and lives in the first quadrant. This makes our life easy and we don't have to change anything. So we have $\arg(z) = \theta =$ ___.

Example 4.4 Now let's consider $z = -1 - i$. Calculate its modulus and argument.

For the modulus we have:

$$|z| =$$

For the argument we have:

$$\theta =$$

Now we must think about the point associated to z. This point is the point __________________ and lives in the third quadrant. This means that we need to alter θ in order to get the argument. In particular, we just need to add π. So we have $\arg(z) = \theta + \pi =$ _____.

Example 4.5 Let's use $z = -4 + 3i$ from our previous examples. Calculate its modulus and argument.

First, we have:

$$|z| =$$

To calculate the argument we have:

$$\theta =$$

Now we must think about the point associated to z. This point is the point _______________ and lives in the second quadrant. This means that we need to alter θ in order to get the argument. In particular, we just need to add π. So we have $\arg(z) = \theta + \pi \approx$ ___________.

We now have enough in order to define our three new forms.

Definition 4.6 Let $z = x + iy$ be a complex number. Let $r = |z|$ be its modulus and $\theta = \arg(z)$ be its argument.

Polar: The *polar form* of z is given by $z = r_\theta$.

Trigonometric: The *trigonometric form* of z is given by $z = r(\cos(\theta) + i\sin(\theta))$.

Exponential: The *exponential form* of z is given by $z = re^{i\theta}$.

Note that the trigonometric form gives us a way to go from r and θ to x and y. In particular,

$$x = \underline{\hspace{3cm}} \quad \textit{and} \quad y = \underline{\hspace{3cm}}.$$

Example 4.7 In small groups, fill out the following table:

Cartesian	Binomial	Polar	Trigonometric	Exponential
$(1, -1)$				
	$\sqrt{3} - i$			
		$3_{\frac{\pi}{4}}$		
			$2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$	
				$4e^{i\pi}$

5 Operations

Just like with the real numbers, there's a lot of different operations we can do with complex numbers.

5.1 Standard operations

The four standard operations (addition, subtraction, multiplication and division) work exactly as one would expect. The only thing to do is you pretend that i is a variable and whenever you have i^2 you replace it with -1.

For addition and subtraction, things are very easy:

- $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
- $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

For multiplication, things start becoming a little more complicated.

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

As one can imagine, division is much more complicated:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \\ &= \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{(x_1x_2 - y_1(-y_2)) + i(x_1(-y_2) + y_1(x_2))}{(x_2x_2 - y_2(-y_2)) + i(x_2y_2 + x_2(-y_2))} \\ &= \frac{(x_1x_2 + y_1y_2) + i(x_1 - y_2 + y_1x_2)}{x_2^2 + y_2^2}\end{aligned}$$

Example 5.1 Let's look at an example. Let $z_1 = 1+i$ and $z_2 = 1-2i$. Calculate the following:

- $z_1 + z_2 =$ ________
- $z_1 - z_2 =$ ____
- $z_1 \cdot z_2 =$ ________
- $\frac{z_1}{z_2} =$ ________________

5.2 Conjugate

The *conjugate* of a complex number $z = x+iy$ is the number $\overline{z} = x-iy$. Notice that all we're doing is changing the sign of the imaginary part.

Wikipedia: conjugate

Proposition 5.2 *Let z, z_1 and z_2 be two complex numbers. Then:*

- $\overline{\overline{z}} = z$
- $z = \overline{z} \iff z \in \mathbb{R}$
- $|z| = |\overline{z}|$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
- $z \cdot \overline{z} = |z|^2$

5.3 Powers and roots

We can also consider powers of complex numbers. This can be seen right away by looking at the powers of i.

$$i^k = \begin{cases} 1 & \text{if } k = 0,\, 4,\, 8,\, 12,\, \ldots\ (k = 4n,\, n \in \mathbb{N}) \\ i & \text{if } k = 1,\, 5,\, 9,\, 13,\, \ldots\ (k = 4n+1,\, n \in \mathbb{N}) \\ -1 & \text{if } k = 2,\, 6,\, 10,\, 14,\, \ldots\ (k = 4n+2,\, n \in \mathbb{N}) \\ -i & \text{if } k = 3,\, 11,\, 15,\, \ldots\ (k = 4n+3,\, n \in \mathbb{N}) \end{cases}$$

They go in a loop!

We can also get the n-th power of an arbitrary complex number $z = re^{i\theta}$ by using its exponential notation.

$$z^n = r^n e^{in\theta}.$$

We can also use the trigonometric form thanks to De Moivre's formula.

Theorem 5.3 (De Moivre's Formula)

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

What about polar form? We have the following operations given $z_1 = r_\alpha$ and $z_2 = s_\beta$:

- $r_\alpha \cdot s_\beta = (r \cdot s)_{\alpha+\beta}$
- $(r_\alpha)^n = r^n_{n\alpha}$
- $\frac{r_\alpha}{s_\beta} = \left(\frac{r}{s}\right)_{\alpha-\beta}$

Example 5.4 For $z = 1 + i$, find z^{12}.

Example 5.5 Let's try another example. Find $i^{2019} + i^{2020} + i^{2021}$.

$$2019 = \underline{\hspace{6cm}}$$
$$2020 = \underline{\hspace{3.5cm}}$$
$$2021 = \underline{\hspace{6cm}}$$

By our formulas in the beginning of this section, this means:

$$i^{2019} = \underline{\hspace{1.5cm}} \qquad i^{2020} = \underline{\hspace{0.5cm}} \qquad i^{2021} = \underline{\hspace{0.3cm}}$$

Putting this together we have:

$$i^{2019} + i^{2020} + i^{2021} =$$

We're finally ready to handle roots. Suppose that z and w are complex numbers and that $n \in \mathbb{N}$. Suppose further that we have $z = w^n$. In other words, w is the n-th root of z. We can rewrite this as $w = z^{\frac{1}{n}} = \sqrt[n]{z}$.

If $z = re^{i\theta} \neq 0$ then its roots are given by:

$$w_k = \sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}}, \quad \text{with } k = 0, 1, 2, \ldots, n-1$$

This tells us that there are n roots! In particular, the n roots are:

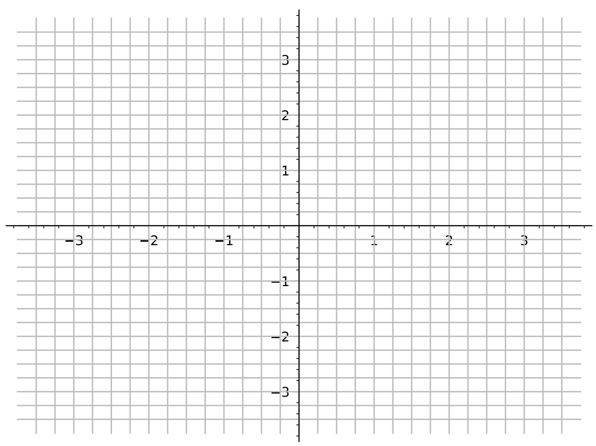
$$\begin{aligned} w_0 &= \sqrt[n]{r}e^{i\theta}n \\ w_1 &= \sqrt[n]{r}e^{i\theta+2\pi}n \\ w_2 &= \sqrt[n]{r}e^{i\theta+4\pi}n \\ &\vdots \\ w_{n-1} &= \sqrt[n]{r}e^{i\theta+2(n-1)\pi}n \end{aligned}$$

The n-th roots of z form a regular polygon with n vertices which are equidistant from the origin with radius $\sqrt[n]{r}$.

Example 5.6 Calculate the cube roots of $z = 3 + 3i$.

Example 5.7 Let's see the polygon using an example. Solve the equation $z^4 - 81 = 0$ for $z \in \mathbb{C}$.

Plotting these points, we get:



Chapter 2

Matrices and Elementary transformations

6 Matrices - Quick Review

Before going into our elementary transformations, we start by giving an ultra quick reminder of the basics of matrices.

Definition 6.1 An $m \times n$ *matrix* is a rectangular grid with m rows and n columns.

Wikipedia: matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \qquad 2 \times 3$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}) \qquad m \times n$$

The following are definitions of things you should have seen.

- *Null matrix*: is the all 0 matrix denoted by $0_{m \times n}$.
- *Square matrix*: a matrix of size $n \times n$.
- *Main diagonal*: the entries a_{ii} of a square matrix.
- *Identity matrix*: is a square matrix with 1 on the main diagonal and 0 everywhere else, denoted by I_n.

6.1 Operations

Recall the following operations on matrices. In particular, assume that A and B are both $m \times n$ matrices and α is some number.

Addition: $A + B = C$ implies $c_{ij} = a_{ij} + b_{ij}$.

Scalar multiplication: $\alpha A = C$ implies $c_{ij} = \alpha * a_{ij}$.

Proposition 6.2 *Let A, B and C be $m \times n$ matrices and α, β be some numbers. Then*

(1) $A + B = B + A$ *(Commutativity of addition)*

(2) $(A + B) + C = A + (B + C)$ *(Associativity of addition)*

(3) $A + 0_{m\times n} = 0_{m\times n} + A = A$ *(Identity element for sum.)*

(4) $A + (-A) = 0_{m\times n}$ *(Opposite matrix)*

(5) $1A = A$ *(Scalar multiplication by unit)*

(6) $(\alpha + \beta)A = \alpha A + \beta A$ *(Distributivity)*

(7) $\alpha(A + B) = \alpha A + \alpha B$ *(Distributivity)*

(8) $\alpha(\beta A) = (\alpha\beta)A$ *(Associativity of product by a scalar)*

Wikipedia: Matrix multiplication

Remember that multiplication is significantly harder to define. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Then

Matrix multiplication was first described by French mathematician Jacques Philippe Marie Binet in 1812, but was formalized in the way we learn it now by Arthur Cayley in 1857.

$$
\begin{aligned}
AB &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{ip} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} \\
&= \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix} \\
&= C
\end{aligned}
$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Example 6.3

$$\begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + -5 \cdot 0 & 1 \cdot 3 + -5 \cdot 1 \\ 2 \cdot 2 + 3 \cdot 0 & 2 \cdot 3 + 3 \cdot 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -2 \\ 4 & 9 \end{pmatrix}$$

Proposition 6.4 *Let A be an $m \times n$ matrix, B and B' be two $n \times p$ matrices and C be a $p \times q$ matrix. Further, let α be a scalar.*

(1) $(AB)C = A(BC)$ (Associativity of multiplication)

(2) $AI_n = I_m A = A$ (Neutral element for multiplication)

(3) $A0_{n\times r} = 0_{m\times r}$ and $0_{s\times m}A = 0_{s\times n}$ (Absorbing element)

(4) $\alpha(AB) = (\alpha A)B = A(\alpha B)$ (Associativity by scalar)

(5) $A(B+B') = AB+AB'$ and $(B+B')C = BC+B'C$ (Distributivity)

Recall also that square matrices can also have determinants (denoted $|A|$) and inverses (denoted A^{-1}). We'll see how to calculate these later on.

7 Types of transformations

There are certain operations we can do in order alter matrices.

7.1 Row operations

Swap two rows of a matrix For a matrix A, to swap the ith and jth row, we can multiply on the left by the following matrix:

$$F_{i,j} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Example 7.1 Swap the rows of the following matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Multiplying a line by a scalar λ If we want to multiply only one row by a scalar λ we can multiply on the left by the following matrix:

$$F_i(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Example 7.2 Make the second row 10 times bigger

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Change a row for the sum of the row and a scalar multiple of another one If we want to take the ith row and add a scalar multiple of the jth row to it, we would multiply on the left by the following matrix:

$$F_{ij}(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & \lambda & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

where the λ is in the ith row and jth column.

Example 7.3 Add 2 times the second row to the first row

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

7.2 Column operations

Column operations are almost identical to row operations, except we multiply on the right instead of on the left.

Swap two columns of a matrix For a matrix A, to swap the ith and jth column, we can multiply on the right by the following matrix:

$$C_{i,j} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Example 7.4 Swap the columns of the following matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Multiplying a column by λ If we want to multiply only one column by a scalar λ we can multiply on the right by the following matrix:

$$C_i(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Example 7.5 Make the second column 10 times bigger

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Change a column for the sum of the column and a scalar multiple of another one If we want to take the ith column and add a scalar multiple of the jth column to it, we would multiply on the right by the following matrix:

$$C_{ij}(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & \lambda & & \ddots & \\ & & & & 1 \end{pmatrix}$$

where the λ is in the jth row and ith column.

Example 7.6 Add 2 times the second column to the first column

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Example 7.7 Here's some examples of some larger transformations.

$$F_{23} = \qquad\qquad F_1(3) =$$

$$F_{13}(5) = \qquad\qquad F_{23}(-1) =$$

7.3 Properties of transformations

We have the following properties:

- $F_{ij} = F_{ji} = C_{ij} = C_{ji}$
- $F_i(\lambda) = C_i(\lambda)$
- $F_{ij}(\lambda) = C_{ji}(\lambda)$
- $|F_{ij}| = |C_{ij}| = -1$
- $|F_i(\lambda)| = |C_i(\lambda)| = \lambda$
- $|F_{ij}(\lambda)| = |C_{ij}(\lambda)| = 1$
- $F_{ij}^{-1} = F_{ji}$
- $C_{ij}^{-1} = C_{ji}$
- $(F_i(\lambda))^{-1} = F_i\left(\frac{1}{\lambda}\right)$
- $(C_i(\lambda))^{-1} = C_i\left(\frac{1}{\lambda}\right)$
- $(F_{ij}(\lambda))^{-1} = F_{ij}(-\lambda)$
- $(C_{ij}(\lambda))^{-1} = C_{ij}(-\lambda)$

8 Matrix properties

Definition 8.1 An $m \times n$ matrix A is said to be *reduced* if it is of the form:

$$\begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}$$

For every $m \times n$ matrix A there exist $r \in \mathbb{N}$, row transformations $F_1, \ldots, F_k$ and column transformations $C_1, \ldots, C_\ell$ such that

$$F_k \cdots F_1 \cdot A \cdot C_1 \cdots C_\ell = \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}$$

The reduced matrix on the right is said to be the *reduced form* of the matrix A. The *rank* of the matrix A is said to be r. If A is a square $n \times n$ matrix, then if $r \neq n$ then $|A| = 0$ otherwise, $r = n$ and we have

$$|A| = \frac{1}{|F_1| \cdots |F_k| \cdot |C_1| \cdots |C_\ell|}$$

Example 8.2 Find the reduced form and determinant of the matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

Example 8.3 Find the reduced form and calculate the determinant of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

8.1 Finding the inverse

We can use row and column operations to find inverses! For any square matrix A whose determinant is not 0 ($|A| \neq 0$), then we can reduce A to a reduced matrix using *only* row transformations, or we can reduce A using *only* column transformations. This is really strong because by doing it this way, we can easily obtain the inverse of A.

How does this work? Well if we only use row transformations, then we know

$$F_k \cdots F_1 \cdot A = I_n$$

This means that $A^{-1} = F_k \cdot F_1$.

In a similar manner, if we only use column transformations, we have $A \cdot C_1 \cdots C_\ell = I_n$ implying $A^{-1} = C_1 \cdots C_\ell$.

A simple trick to calculate the inverse is to append the identity matrix to the end and to just do our transformations like we normally would. Let's look at our previous example to see how this would work.

Example 8.4 We start with appending the identity matrix to the end of A. We put a bar between the two matrices so that we know we are working with two different matrices at the same time.

$$A = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$F_{31}(-1) \cdot A = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$F_{23}(-1) \cdot F_{13}(1) \cdot F_{31}(-1) \cdot A = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$F_2\left(\frac{1}{2}\right) \cdot F_{23}(-1) \cdot F_{13}(1) \cdot F_{31}(-1) \cdot A = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

This means that the matrix on the right is our inverse matrix. In other words,

$$A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \end{pmatrix}$$

To double-check, just multiply A and A^{-1} to make sure you get I_3.

If we don't want to do all of this, and we already know our row/column transformations, we can just multiply their matrices which we gave in section 7.

Example 8.5 Let's do the same example from before to see this in action. We already know that $F_2\left(\frac{1}{2}\right) \cdot F_{23}(-1) \cdot F_{13}(1) \cdot F_{31}(-1) \cdot A = I_3$ and so our inverse matrix should be $F_2\left(\frac{1}{2}\right) \cdot F_{23}(-1) \cdot F_{13}(1) \cdot F_{31}(-1)$. Let's see this.

$$\begin{aligned} &F_2\left(\frac{1}{2}\right) \cdot F_{23}(-1) \cdot F_{13}(1) \cdot F_{31}(-1) \\ &\quad = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 2 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \end{pmatrix} \end{aligned}$$

Everything we just did has a name and is a very standard way to find the reduced form of a matrix.

Definition 8.6 Let A be a matrix. *Gauss's method* consists in finding row (or column) transformations such that

$$F_n \cdots F_1 \cdot A = U \quad A \cdot C_1 \cdots C_\ell = L$$

where U and L are upper and lower triangular matrices respectively.

Gauss-Jordan elimination consists in finding the reduced form of A using the elementary row (or column) transformations.

Wikipedia: Gauss-Jordan elimination

9 Systems of linear equations

We can use matrices to help us solve systems of linear equations.

Definition 9.1 A *system of linear equations* is a set of equations with n variables. We normally label the variables x_1 up to x_n. As it's linear, no variable is ever multiplied with another variable (so we never have x_ix_j). A system of linear equations normally looks like:

Wikipedia: system of linear equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{cases}$$

Notice how the coefficients look exactly like what we've been doing with matrices! In fact, we can represent this system of equations using matrices. To do this, we set the following three matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

To get back our system of linear equations we just multiply everything together:

$$A \cdot X = B$$

Notice how each row relates to a certain equation from above.

Our main goal is to simultaneously solve for the values of our variables. To do this, we can use our matrices. Just like we did for inverses, we merge two of our matrices together to form what is known as an

Wikipedia: augmented matrix

augmented matrix (denoted $A^\star$). In particular, we let

$$A^\star = \left(A \mid B\right) = \left(\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{array}\right)$$

Wikipedia: Rouché-Capelli Theorem

Theorem 9.2 (Rouché-Capelli Theorem) *Given a system of linear equations $A \cdot X = B$ with m equations and n unknowns. Let $A^\star$ be its augmented matrix. Then:*

- *If $n = \operatorname{rk}(A) = \operatorname{rk}(A^\star)$ then there is a* unique *solution.*
- *If $n > \operatorname{rk}(A) = \operatorname{rk}(A^\star)$ then there are* infinitely many *solutions.*
- *If $\operatorname{rk}(A) \neq \operatorname{rk}(A^\star)$ then* no *solutions exist.*

Note that if $B = 0_{m\times 1}$ (in other words, all equations are equal to 0), then we always have at least one solution since $X = 0_{n\times 1}$ (in other words, setting all our variables equal to 0) is a solution (the trivial solution). This type of system is known as a *homogenous system*.

Recall from the beginning of section 8 that the rank of A is equal to n if and only if $|A| = 0$. This gives us the following corollary.

Corollary 9.3 *Let A be an $n \times n$ square matrix and let X, and B be matrices as above. The following are equivalent.*

- $\operatorname{rk}(A) = n$
- *$A \cdot X = B$ has a unique solution.*
- $|A| \neq 0$

Let's see examples of each of these.

Example 9.4

$$[A|B] = \left(\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 0 & 1 & 5 & 11 \\ 0 & 1 & 6 & 13 \\ -2 & -1 & -11 & -16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

What this tells me is that $\operatorname{rk}(A) = 3$ and $\operatorname{rk}(A^\star) = 4$ and therefore, there is no solution.

Example 9.5

$$[A|B] = \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 1 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the ranks are the same and they are equal to $n = 3$, then we have a unique solution. This is given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \\ 0 \end{pmatrix}$$

implying

$$x_1 = 2 \quad x_2 = -2 \quad x_3 = 3 \quad 0 = 0$$

is our unique solution.

Example 9.6

$$[A|B] = \left(\begin{array}{cccc|c} 1 & 5 & 3 & -11 & 11 \\ 0 & 1 & 1 & -2 & 1 \\ 1 & 6 & 5 & -13 & 10 \\ 1 & 4 & 0 & -9 & 14 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since our rank is 3, but $n = 4$, we have an infinite number of solutions. We see this when we use our matrices.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 0 \end{pmatrix}$$

In other words,

$$x_1 - x_4 = 2 \quad x_2 - 2x_4 = 3 \quad x_3 = -2 \quad 0 = 0$$

From here we get:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 + 2 \\ 2x_4 + 3 \\ -2x_4 \end{pmatrix}$$

Chapter 3

Vector spaces

10 Vector spaces

Definition 10.1 A *vector space* is a triple $(V, +, \cdot, \mathbb{K})$ where

Wikipedia: vector space

- V is a set (we call the objects in the set *vectors*).

Wikipedia: vectors

- $+$ is an operation called *vector addition* such that for all $x, y \in V$ then $x + y \in V$.
- $\cdot$ is an operation called *scalar multiplication* such that for all $x \in V$ and all $\lambda \in \mathbb{K}$ then $\lambda \cdot x \in V$.

Wikipedia: scalar multiplication

- $\mathbb{K}$ is a *field* (can assume this to be $\mathbb{R}$ or $\mathbb{C}$).

Wikipedia: Field

such that the following properties hold:

- for all $x, y \in V$ we have $x + y = y = x$ (Commutativity of vector addition)
- for all $x, y, z \in V$ we have $x + (y + z) = (x + y) + z$ (Associativity of vector addition)
- There exists an element $e \in V$ (called the *zero vector*) such that for all $x \in V$ $x + e = e + x = x$. We normally denote e as 0 (Identity element of vector addition)
- For all $x \in V$ there exists $-x \in V$ such that $x + (-x) = (-x) + x = 0$. (Inverse elements of vector addition)
- For all $x \in V$ and $\lambda, \mu \in \mathbb{K}$ we have $\lambda \cdot (\mu \cdot x) = (\lambda \cdot \mu) \cdot x$. (Associativity of scalar multiplication)
- There exists an element $a \in \mathbb{K}$ such that for all $x \in V$ we have $a \cdot x = x$. We normally denote a as 1 (Identity element of scalar multiplication)

- For all $\lambda \in \mathbb{K}$ and $x, y \in V$ we have $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$. (Distributivity of vector addition)
- For all $\lambda, \mu \in \mathbb{K}$ and $x \in V$ we have $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$. (Distributivity of scalar multiplication)

Example 10.2 As an example, we can take column matrices and use matrix addition and scalar multiplication from chapter 2. We would denote this as ____________________.

What does addition and multiplication look like then? Given the following two vectors:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Then,

$$x + y = \qquad\qquad\qquad \lambda \cdot x =$$

11 Linear combinations

If we think about space around us, how do we calculate the number of dimensions? We normally "add" a dimension in some way, but mathematically, how does that work? We do that through the idea of linear combinations and linear independence.

Wikipedia: linear combination

Definition 11.1 Given a set of vectors $\{x_1, \ldots, x_n\} \in \mathbb{R}^n$ we say that x is a *linear combination* of $\{x_1, \ldots, x_n\}$ if there exists a set of real numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$x = \lambda_1 x_1 + \ldots + \lambda_n x_n = \sum_{i=1}^{n} \lambda_i x_i.$$

Example 11.2 Let's use our vector space of column matrices from before and let $n = 3$. We set the following:

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Write the following vector as a linear combination of $\{x_1, x_2\}$:

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Definition 11.3 We say that a set of vectors $\{x_1, \ldots, x_n\} \in \mathbb{R}^n$ are *linearly dependent* if there exists a set of coefficients $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (not all 0) such that

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = 0$$

In the case where the linear combination is not equal to 0, we say that the vectors are *linearly independent*. Another way of saying this is that a set of vectors are linearly independent if

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = 0 \Rightarrow \lambda_i = 0, \forall i$$

When we're considering our vector space to be the column matrices, then if we recall from section 9 this equation is nothing more than our homogeneous system of linear equations! It turns out that if our set of vectors give us a *unique* solution, then our vectors are linearly independent! Why? ____

____

If on the other hand, our vectors give us *infinitely many* solutions, then the vectors are linearly dependent. Do you remember why we can't have the option that no solution exists? ________________________

____

____

Let's see this in action.

Example 11.4 Suppose we have the following set of vectors. Show that they are linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$$

Example 11.5 Let's try another example using another method. Suppose we have the following three vectors, find λ_i to show they are linearly dependent.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right\}$$

12 Bases

Just a couple more definitions before we're able to define dimension mathematically.

12.1 Spanning sets and bases

Definition 12.1 In $\mathbb{R}^n$, a set of vectors $\{x_1, \dots, x_m\}$ is a *spanning set* if for every $x \in \mathbb{R}^n$, there exist coefficients $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

Wikipedia: spanning set

$$x = \sum_{i=1}^{n} \lambda_i x_i$$

If additionally, the set of vectors are linearly independent, then we say they form a *basis* of $\mathbb{R}^n$.

Wikipedia: basis

The dimension of a vector space is nothing more than the number of vectors in its basis.

Definition 12.2 If $B = \{x_1, \dots, x_n\}$ is a basis of a vector space V, then we say that V has *dimension* n.

$$\dim(V) = \#(B).$$

Example 12.3 Show that the following set of vectors is not a generating set for $\mathbb{R}^2$.

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right\}$$

Example 12.4 Show that the following is a generating set (and basis) for $\mathbb{R}^2$.

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$$

Example 12.5 The following is a generating set for $\mathbb{R}^3$ that is not a basis.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}$$

Example 12.6 The following is a basis for $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

This method, of just removing the vectors that we don't need, is a good way to find bases. It helps us reduce a set of linearly dependent set into a minimally linearly independent set.

Just because it's minimally linearly independent does *not* make it a basis. It's only a basis if it is linearly independent *and* is a generating set for our vector space.

Minimally linearly independent set extraction Given a set of (potentially) linearly dependent vectors $\{x_1, \ldots, x_n\}$. We follow the following steps.

Start with $B = \varnothing$ being the empty set. We do the following on a loop from $i = 1$ until $i = n$. (Starting from $i = 1$ and increasing i by one each time we do the loop)

- Let $b_{i-1} = \mathrm{rk}(B)$.
- Find the rank $b_i = \mathrm{rk}(B \cup \{x_i\})$
- If $b_i > b_{i-1}$ then add x_i into B and repeat the loop.
- If $b_i = b_{i-1}$ then this is a linearly independent set, so we *do not* change B. We repeat the loop without changing anything.

Let's try this on an example.

Example 12.7 Suppose we have the following spanning set which is not a basis of $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} = \{x_1, x_2, x_3, x_4\}$$

Find a basis using the extraction method from above.

12.2 Canonical basis

There is a standard basis that we generally use more than other ones. It's called the canonical basis and is generally the easiest to work with.

Wikipedia: standard basis

Definition 12.8 The *standard basis* (or the *canonical basis*) of $\mathbb{R}^n$ is the basis given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \right\}$$

It's not too hard to see that this is a basis of $\mathbb{R}^n$. We use this basis as the starting point and in the next section we look at how to go from one basis of a vector space to another.

13 Change of bases

Suppose that we have two different bases of a vector space V. Let $B = \{u_1, \dots, u_n\}$ be the first one and $B' = \{u'_1, \dots, u'_n\}$ be the second one. Furthermore, let $x \in \mathbb{R}^n$. By definition of B being a basis, there exist non-zero $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$\begin{aligned}
x &= \lambda_1 u_1 + \ldots + \lambda_n u_n \\
&= (u_1 \cdot u_n) \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \\
&= \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1j} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2j} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ u_{i1} & u_{i2} & \cdots & u_{ij} & \cdots & u_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nj} & \cdots & u_{nn} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \\
&= P \cdot x_B
\end{aligned}$$

What did we just do? Well, the first row is nothing more than the definition of basis: every element of $\mathbb{R}^n$ can be written as a linear combination of the elements of the basis. We then converted this into two matrices (one horizontal and one vertical). Note that performing the multiplication gives us the linear combination. We then noted that each u_i is itself a column vector. So we expanded each u_i into its own vector so that our horizontal matrix became a square matrix. Finally, we gave these two matrices names.

We call P the *matrix associated to the basis B*. And we call x_B the *coordinates of x with respect to the basis B*.

We can do the same thing with our other basis B'. In other words, there exist $\lambda'_1, \ldots, \lambda'_n \in \mathbb{R}$ such that

$$\begin{aligned}
x &= \lambda'_1 u'_1 + \ldots + \lambda'_n u'_n \\
&= (u'_1 \cdot u'_n) \cdot \begin{pmatrix} \lambda'_1 \\ \vdots \\ \lambda'_n \end{pmatrix} \\
&= \begin{pmatrix} u'_{11} & u'_{12} & \cdots & u'_{1j} & \cdots & u'_{1n} \\ u'_{21} & u'_{22} & \cdots & u'_{2j} & \cdots & u'_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ u'_{i1} & u'_{i2} & \cdots & u'_{ij} & \cdots & u'_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ u'_{n1} & u'_{n2} & \cdots & u'_{nj} & \cdots & u'_{nn} \end{pmatrix} \cdot \begin{pmatrix} \lambda'_1 \\ \vdots \\ \lambda'_n \end{pmatrix} \\
&= Q \cdot x_{B'}
\end{aligned}$$

But now notice that since x is equal to both of these, we have

$$P \cdot x_B = x = Q \cdot x_{B'}.$$

But this allows us to send coordinates from one basis to coordinates of another!

Definition 13.1 The *change of basis* from a basis B to a basis B' for an element x is given by:

$$x_{B'} = Q^{-1} \cdot P \cdot x_B$$

Example 13.2 Let's look at an example. Suppose we're working in $\mathbb{R}^3$, and we have the following two bases:

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad B' = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

This gives us the following P and Q:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

We now calculate the inverses. (If you don't remember how to do this, now's a good time to review.)

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad Q^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

This means our change of basis from B to B' is given by:

$$x_{B'} =$$

The change of basis from B' to B is then given by:

$$x_B =$$

Now, let's actually see what happens with an element of $\mathbb{R}^3$ when we use the change of basis. Let $x \in \mathbb{R}^3$ be the following element

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Calculate x_B and $x_{B'}$.

We can now use this information to solve for $x_{B'}$.

Let's try another one. Suppose that we have $y_{B'} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Find y_B and y.

Notice that both x and y are themselves column matrices. What basis are we using when we write them out? ____________________

14 Linear subspaces

14.1 Linear subspace

Wikipedia: linear subspace

Definition 14.1 Let $\mathbb{R}^n$ be our vector space. A *linear subspace* L is a subset of vectors $L \subseteq \mathbb{R}^n$ such that the following two properties are satisfied:

- $\forall x, y \in L$, $x + y \in L$
- $\forall x \in L$, $\lambda \in \mathbb{R}$, $\lambda \cdot x \in L$.

The reason we call this a "subspace" is because if L is a linear subspace of a vector space V, then L itself is a vector space. This is not too hard to see as commutativity, associativity, the identity element of scalar multiplication and distributivity all come from V. The identity element of vector addition comes from $0 \cdot x = e$ (where $0 \in \mathbb{R}$ and $e \in V$ is the identity element of vector addition). Finally, to find the inverse element of vector addition, we recall that $-1 \in \mathbb{R}$ and so $x + (-1)x = x - x = e$ gives us our inverse.

Example 14.2 Show that the following set is a linear subspace.

$$L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ \middle|\ y = 2x \right\}$$

Example 14.3 Show that the following set is *not* a linear subspace.

$$M = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 3x - 1 \right\}$$

Since a linear subspace must be closed under vector addition and scalar multiplication, we can generate a linear subspace from some subset of vectors of $\mathbb{R}^n$.

Definition 14.4 Given a subset $I = \{a_1, \ldots, a_k\} \subseteq \mathbb{R}^n$ (we assume $k < n$) we let $\langle I \rangle$ denote the *linear subspace generated by I* where $\langle I \rangle$ is the set of all linear combinations of I. Note that sometimes $\mathcal{L}\langle I \rangle$ is used to denote this linear subspace.

$$\langle I \rangle = \left\{ \sum_{i=1}^{k} \lambda_i a_k \;\middle|\; \lambda_i \in \mathbb{R} \right\}$$

Given a linear subspace $L = \langle I \rangle$ generated by some subset $I \subseteq \mathbb{R}^n$, since L is a vector space, we want to find a basis for L. For this, we set some definitions.

Definition 14.5 Given $B = \{a_1, \ldots, a_k\}$ a basis for a linear subspace $L \subseteq \mathbb{R}^n$. We have the following three definitions:

- **Linear equation**

$$x = \lambda_1 a_1 + \ldots + \lambda_k a_k$$

- **Parametric equations**

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix} + \ldots + \lambda_k \begin{pmatrix} a_{k1} \\ \vdots \\ a_{kn} \end{pmatrix}$$

Giving us the set of equations

$$\begin{cases} x_1 & = & \lambda_1 a_{11} + \ldots + \lambda_k a_{k1} \\ \vdots & \vdots & \vdots \\ x_n & = & \lambda_1 a_{1n} + \ldots + \lambda_k a_{kn} \end{cases}$$

- **Implicit equations** For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in L$, we impose:

$$\operatorname{rk} \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{k1} & x_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & \cdots & a_{kn} & x_n \end{array}\right) = k$$

Example 14.6 Let's see each of these in action by calculating the basis, equations and dimension. We start with a linear subspace generated by a set.

$$L = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ 2 \end{pmatrix} \right\rangle$$

Basis and dimension:

Equations:

From our implicit equations, we have the following theorem.

Proposition 14.7 *Let $L \subseteq \mathbb{R}^n$ be a linear subspace. Then*

$$\dim(L) = n - \#\{\textit{linearly independent implicit equations}\}$$

Operations

We now define some operations on linear subspaces.

Definition 14.8 Given two linear subspaces L_1 and L_2 of $\mathbb{R}^n$, we define two linear subspaces:

- $L_1 + L_2 = \{x \in \mathbb{R}^n \mid x = x_1 + x_2,\, x_1 \in L_1,\, x_2 \in L_2\}$
- $L_1 \cap L_2 = \{x \in \mathbb{R}^n \mid x \in L_1,\, x \in L_2\}$

We call the first linear subspace the *sum* of linear subspaces and the second linear subspace the *intersection* of linear subspaces.

We can use these two operations to define the complement of a linear subspace.

Definition 14.9 The *complement* of a linear subspace is the linear subspace $L^\star$ such that:

$$L + L^\star = \mathbb{R}^n \quad \text{and} \quad L \cap L^\star = \{0\}$$

Since we can represent this in different ways, we define a new operation to keep track of when the above happens.

Definition 14.10 We say two linear subspaces L_1 and L_2 form a *direct sum* (denotes $L_1 \oplus L_2$) if:

$$L_1 + L_2 = \mathbb{R}^n \quad \text{and} \quad L_1 \cap L_2 = \{0\}$$

Notice that a linear subspace and its complement always form a direct sum.

Proposition 14.11 *Given two linear subspaces $L_1, L_2 \subseteq \mathbb{R}^n$ whose bases are given by $B_{L_1} = \{a_1, \ldots, a_i\}$ and $B_{L_2} = \{b_1, \ldots, b_j\}$ respectively. Additionally, suppose we have the following system of implicit equations for L_1 and L_2 respectively:*

$$L_1 : \left\{ \begin{array}{rcl} a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{r1}x_1 + \ldots + a_{rn}x_n & = & 0 \end{array} \right. \qquad L_2 : \left\{ \begin{array}{rcl} b_{11}x_1 + \ldots + b_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ b_{s1}x_1 + \ldots + b_{sn}x_n & = & 0 \end{array} \right.$$

Then, $B = B_{L_1} \cup B_{L_2} = \{a_1, \ldots, a_i, b_1, \ldots, b_j\}$ is a generating set for $L_1 + L_2$. Furthermore, the implicit equations of $L_1 \cap L_2$ are given by:

$$\left\{ \begin{array}{rcl} a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{r1}x_1 + \ldots + a_{rn}x_n & = & 0 \\ b_{11}x_1 + \ldots + b_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ b_{s1}x_1 + \ldots + b_{sn}x_n & = & 0 \end{array} \right.$$

To find $B_{L_1^\star}$ we complete B_{L_1} until we have a basis of $\mathbb{R}^n$.

Example 14.12 Let's look at a large example in $\mathbb{R}^4$. Consider the following two linear subspaces, defined in different ways.

$$F : \begin{cases} x - t &= 0 \\ y + z &= 0 \end{cases} \qquad G = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

Find the basis, dimension and equations of:

$$F,\ G,\ F + G,\ F \cap G,\ F^\star,\ G^\star$$

F**:**

G**:**

$F + G$**:**

$F \cap G$**:**

$F^{\star}$**:**

$G^{\star}$**:**

Chapter 4

Linear maps

15 Associated matrix

In the last chapter, we started working with vector spaces, but what happens if we want to define a map from one vector space to another? Since we're living in the world of vector spaces, we want these maps to function correctly in regards to vector addition and scalar multiplication. This gives us what is known as a linear map.

Definition 15.1 A *linear map* is a map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that

Wikipedia: linear map

(1) $f(u+v) = f(u) + f(v)$, for all $u, v \in \mathbb{R}^n$.

(2) $f(\lambda u) = \lambda f(u)$, for all $\lambda \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$.

Let's look at a couple examples.

Example 15.2 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the following map:

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

Show that it is linear.

Example 15.3 Show that the following map $f : \mathbb{R}^3 \to \mathbb{R}^2$ is *not* linear.

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + y + z^2 \\ x - z \end{pmatrix}$$

Since linear maps allow us to go from one vector space to another, we can define a sort of change of basis: a matrix that is associated to the linear map, sending us from one vector space to another.

Definition 15.4 Given a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ whose vector spaces have bases $B = \{b_1, \ldots, b_m\}$ and $C = \{c_1, \ldots, c_m\}$ respectively. Given an element $x \in \mathbb{R}^n$ there exists $x_1, \ldots, x_n \in \mathbb{R}$ such that $x = x_1 b_1 + \ldots + x_n b_n$. We say that $x_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ are *the coordinates of x with respect to B*.

Applying f we have the following:

$$f(x) = f(x_1 b_1 + \ldots + x_n b_n) = x_1 f(b_1) + \ldots + x_n f(b_n)$$

with each $f(b_i) \in \mathbb{R}^m$. Additionally, for each $f(b_i)$ there exists $\alpha_{i1}, \ldots \alpha_{im} \in \mathbb{R}$ such that $f(b_i) = \alpha_{i1} c_1 + \ldots + \alpha_{im} c_m$ for all $i = 1, \ldots n$. The coordinates of $f(b_i)$ with respect to the basis C are then $(f(b_i))_C = \begin{pmatrix} \alpha_{i1} \\ \vdots \\ \alpha_{im} \end{pmatrix}$.

This gives us the following:

$$(f(x))_C = ((f(b_1))_C \cdots (f(b_n))_C) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \cdot x_B$$

The matrix A is known as the *matrix of f with respect to B and C*.

Let's see this in action in the following example.

Example 15.5 Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map given by the following with respect to the canonical basis:

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ -x \\ -x + y \end{pmatrix}$$

Consider the following additional bases:

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\} \qquad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$B' = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \qquad C' = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \right\}$$

(1) Calculate the matrix of f with respect to the canonical bases.

(2) Give the matrix expression of f with respect to B and C.

(3) With a partner, give the matrix expression of f with respect to B' and C'.

(4) Calculate $(f(x_{B'}))_{C'}$ when $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(5) Calculate $(f(x_{B'}))_{C}$ when $x_{B'} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

16 Matrix expression and change of basis

What happens if we need to change vector spaces multiple times? Just like with functions, we can compose linear maps.

Given two linear maps $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ whose associated matrices are given by A and B respectively. The *composition* of g with f, $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$, given by $(g \circ f)(x) = g(f(x))$ has the associated matrix $C = B \cdot A$.

This makes jumping vector spaces using linear maps extremely nice and easy. In fact, this multiplication extends to changing of bases since a change of basis is nothing more than a linear map from a vector space to itself.

Given a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$, two bases B and B' of $\mathbb{R}^n$ and two bases C and C' of $\mathbb{R}^m$. Let A be the matrix associated to the linear map f with respect to B and C:

$$(f(X_B))_C = A \cdot X_B$$

Is there a nice way to "easily" calculate the matrix M associated to f with respect to B' and C'?

$$(f(X_{B'}))_{C'} = M \cdot X_{B'}$$

We have the following maps to help us:

$$\begin{tikzcd}
(\mathbb{R}^n, B) \arrow[r, "f"] & (\mathbb{R}^m, C) \\[-20pt]
x_B \arrow[r, mapsto, "A"] & x_C \arrow[d] \\
(\mathbb{R}^n, B') \arrow[u] \arrow[r, "f"] & (\mathbb{R}^m, C') \\[-20pt]
x_{B'} \arrow[r, mapsto, "M"] & x_{C'}
\end{tikzcd}$$

So let's figure this out. Let P, Q, R and S be the matrices associated with B, B', C and C' respectively. From earlier we have the following change of bases:

$$P \cdot X_B = Q \cdot X_{B'} \qquad R \cdot (f(X_B))_C = S \cdot (f(X_{B'}))_{C'}$$

Implying,

$$X_B = P^{-1} \cdot Q \cdot X_{B'} \qquad S^{-1} \cdot R \cdot (f(X_B))_C = (f(X_{B'}))_{C'}$$

And since $(f(X_B))_C = A \cdot X_B$, we obtain:

$$(f(X_{B'}))_{C'} = S^{-1} \cdot R \cdot A \cdot X_B = S^{-1} \cdot R \cdot A \cdot P^{-1} \cdot Q \cdot X_{B'}$$

This gives us the value for M we were trying to find:

$$M = S^{-1} \cdot R \cdot A \cdot P^{-1} \cdot Q$$

16.1 Canonical bases

But what happens we consider our original bases the canonical bases?

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with associated matrix A with respect to the canonical bases. Let B and C be two other bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Furthermore, we let P and Q be the matrices associated to B and C respectively. We have

$$I_n \cdot X = P \cdot X_B \quad I_m \cdot f(X) = Q \cdot (f(X_B))_C \quad f(X) = A \cdot X$$

Therefore,

$$X = P \cdot X_B \quad \text{and} (f(X_B))_C = Q^{-1} \cdot f(X).$$

Implying,

$$(f(X_B))_C = Q^{-1} \cdot A \cdot X = Q^{-1} \cdot A \cdot P \cdot X_B.$$

In other words, we have $M = Q^{-1} \cdot A \cdot P$ in this case.

Example 16.1 Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map given by:

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

with respect to the canonical bases. Calculate the matrix expression of f with respect to the bases:

$$B = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We next give some definitions in regards to linear maps and matrices which we use later on.

Definition 16.2 Linear maps are sometimes called *homomorphisms* of vector spaces. When the homomorphism goes from the vector space to itself, then it's called an *endomorphism* : $f : \mathbb{R}^n \to \mathbb{R}^n$.

Wikipedia: homomorphisms
Wikipedia: endomorphism

Definition 16.3 Given two matrices A and B, we say that they are *similar* if there exists an invertible matrix P such that

Wikipedia: similar

$$B = P^{-1} \cdot A \cdot P.$$

Similar matrices show up whenever we look at endomorphisms. In fact, the matrices associated to an endomorphism under a change of basis are similar: $M = P^{-1} \cdot A \cdot P$.

Similar matrices are nice because they give us the following two properties for free. Assume A and B are similar matrices.

- $\mathrm{rk}(A) = \mathrm{rk}(B)$
- $|A| = |B|$ if A and B are square matrices.

17 Image and kernel

Definition 17.1 Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The *image* of f is the set of vectors in $\mathbb{R}^m$ reachable by f:

Wikipedia: image

$$\mathrm{Im}(f) = \{y \in \mathbb{R}^m \mid y = f(x), x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

The *kernel* of f is the set of elements which get sent to 0:

Wikipedia: kernel

$$\mathrm{Ker}(f) = \{x \in \mathbb{R}^n \mid f(x) = 0 \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

These two objects are related to one another in a very natural way as can be seen in the following proposition and theorem.

Proposition 17.2 *If $L = \{a_1, \ldots, a_k\}$ is a generating set of $\mathbb{R}^n$, then $\{f(a_1), \ldots, f(a_k)\}$ is a generating set for $\mathrm{Im}(f)$. In other words, if $f(X) = M \cdot X$, then the columns of M is a generating set of $\mathrm{Im}(f)$.*

Theorem 17.3 *Given a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$. Then*

$$\dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f)) = n$$

Example 17.4 Let's look at an example of this. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map such that

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Calculate the bases, dimension and matrix equations for $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

18 Image and preimage of a subspace

Definition 18.1 Given a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ and subspaces $L \subseteq \mathbb{R}^n$, $M \subseteq \mathbb{R}^m$, we define the following:

- the *image of L with respect to f* as the set:

$$f(L) = \{y \in \mathbb{R}^m \mid y = f(x), x \in L\}$$

- the *preimage of M with respect to f* as the set:

$$f^{-1}(M) = \{x \in \mathbb{R}^n \mid f(x) \in M\}$$

Note that if $L = \langle a_1, \ldots, a_k \rangle$ then $f(L) = \langle f(a_1), \ldots, f(a_k) \rangle$. If $B_L = \{a_1, \ldots, a_\ell\}$ with $\ell \leq k$, then $\{f(a_1), \ldots, f(a_\ell)\}$ is a *generating set* of $f(L)$. If $M : \{Q \cdot X = 0$, then $f^{-1}(M) : \{Q \cdot A \cdot X = 0$,

Example 18.2 Given a linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that:

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Given the subspaces given by:

$$L : \left\{ x - y \quad = 0 \subseteq \mathbb{R}^2 \quad \text{and} M = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \subseteq \mathbb{R}^3.\right.$$

Calculate the basis, dimension and equations of $f(L)$ and $f^{-1}(M)$.

19 Characteristic polynomial, eigenvalues and eigenvectors

Suppose we have some endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ where B is a basis of $\mathbb{R}^n$ and A is the square matrix of order n associated to f with respect to B. What we want to do is find a basis C of $\mathbb{R}^n$ such that the matrix associated to f with respect to C is a diagonal matrix. Why? Because diagonal matrices are super nice, calculating the determinant is very easy, and it makes our lives much easier.

We look at three different equations that are all related to one another:

$$f(X) = (\lambda I_n) \cdot X \qquad A \cdot X = (\lambda I_n) \cdot X \qquad (A - \lambda I_n) \cdot X = 0$$

We want to find (non-trivial) solutions of the previous equation. In particular, we want to see when $A - \lambda I$ has determinant 0. There's one simple way for this to happen:

$$\text{rk}(A - \lambda I_n) < n \Rightarrow |A - \lambda I_n| = 0$$

In order to work with these equations, we define a bunch of things.

Wikipedia: characteristic polynomial

Definition 19.1
- The *characteristic polynomial* of A is the polynomial $p(\lambda) = |A - \lambda I_n|$.
- The *characteristic equation* of A is given by $p(\lambda) = 0$.
- The *eigenvalues* of A are the roots of the characteristic polynomial, *i.e.*, the solutions of the characteristic equation. We normally denote these as: $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- The *eigenvectors* associated to an eigenvalue λ_i are the solutions to the system $(A - \lambda_i I_n) \cdot X = 0$.
- The *invariant subspace* V_λ associated to an eigenvalue λ is the set of eigenvectors associated with λ.

Wikipedia: eigenvalues

Wikipedia: invariant subspace

Since a lot of things have to do with these eigenvectors, we want an easier way to calculate them. The following proposition gives us one of these methods.

Proposition 19.2 *A vector v is an eigenvector associated to an eigenvalue λ if and only if*

$$f(v) = \lambda v \quad \textit{in other words:} \quad Av = \lambda v$$

Example 19.3 Calculate the characteristic polynomial, eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

Let's focus a little on the characteristic polynomial and try to determine the coefficients.

$$|A - \lambda I_n| = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

We do this using an example.

Example 19.4 Calculate the characteristic polynomial, eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} -2 & 4 & 5 \\ -3 & 5 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

We start with the characteristic polynomial and look at the coefficients.

We next calculate the eigenvectors.

20 Diagonalization

As we can see, the main diagonal tells us a lot of information when figuring out the eigenvalues and eigenvectors. But, there's an even easier way to calculate these when we can convert A to a diagonal matrix.

Wikipedia: diagonalizable

Definition 20.1 We say that a matrix is *diagonal* if the only non-zero entries are in the main diagonal. We say that a matrix A is *diagonalizable* if it is similar to a diagonal matrix. In other words, there exists an invertible matrix P and a diagonal matrix M such that $P^{-1}AP = D$.

It turns out that an endomorphism f of $\mathbb{R}^n$ is diagonalizable if its associated matrix is diagonalizable! But when is a matrix diagonalizable? Before we mention when, we need two definitions.

Definition 20.2 Let A be a matrix whose characteristic polynomial is given by

$$p(\lambda) = (\lambda - \lambda_1)^{\alpha_1}(\lambda - \lambda_2)^{\alpha_2} \cdots (\lambda - \lambda_m)^{\alpha_m}$$

- The *algebraic multiplicity* of an eigenvalue λ_i is the exponent α_i.
- The *geometric multiplicity* of an eigenvalue λ_i is $\dim(V_{\lambda_i})$.

Theorem 20.3 *A matrix A is diagonalizable if for every eigenvalue, its algebraic and geometric multiplicities coincide.*

Ok, but why do we care? Because our P and D actually tell us all the information we could want about eigenvalues and eigenvectors. If A is diagonalizable with $D = P^{-1}AP$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $v_1, \ldots, v_n$ then:

$$P = (v_1 \cdots v_n) \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$$

This tells us so much! We have the following properties.

Proposition 20.4 *Let A be a square matrix of order n.*

- *If A has n distinct eigenvalues, then A is diagonalizable.*
- $\text{tr}(A) = \sum_i \lambda_i$.
- $|A| = \Pi_i \lambda_i$.

The inverse of the first item is *not* true. In particular, if A is diagonalizable, then it doesn't necessarily have n distinct eigenvalues. We see this in the following example.

Let's look at Example 19.4 from this point of view.

Example 20.5 Recall that we are working with the matrix

$$A = \begin{pmatrix} -2 & 4 & 5 \\ -3 & 5 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 20.6 Let's do one additional example. Analyze the following matrix and determine if it's diagonalizable. If so, find P and D.

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 0 & 1 & 0 \\ -2 & 0 & -3 \end{pmatrix}$$

Chapter 5

Euclidean space

21 Inner product and Euclidean space

When thinking about vectors we have two main operations we've been using: vector addition and scalar multiplication. It turns out, there's another operation we can define on vectors called the inner product. It's a form of multiplying to vectors together. In order to not confuse ourselves by using $\cdot$ (which is mainly being used for matrix multiplication), we'll use a different notation for the inner product.

Definition 21.1 In $\mathbb{R}^n$, the *inner product* is a map $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that:

Wikipedia: inner product

- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, $\forall x, y, z \in \mathbb{R}^n$
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, $\langle x, \lambda y \rangle = \lambda \langle x, y \rangle$, $\forall x, y \in \mathbb{R}^n$ $\forall \lambda \in \mathbb{R}$.
- $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0$ if and only if $x = 0$; $\forall x \in \mathbb{R}^n$.
- $\langle x, y \rangle = \langle y, x \rangle$, $\forall x, y \in \mathbb{R}^n$.

A real vector space $\mathbb{R}^n$ which has an inner product is known as a *Euclidean vector space* .

Wikipedia: Euclidean vector space

When working with the Euclidean vector space, it is typical to define a particular inner product which is called the *dot product* . The dot product is the inner product defined as:

Wikipedia: dot product

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 y_1 + \ldots + x_n y_n$$

Example 21.2 As a quick example in $\mathbb{R}^4$, we calculate the dot product given the following two vectors:

$$x = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \quad y = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

22 Norms, angles, distance and orthogonality

When we think about Euclidean space, we also normally think of how far apart two points are or the angle between two lines. We can replicate all of this in Euclidean vector spaces! Let's build our way up to each of these terms. We first define the norm which is a way to define the length of a vector.

Wikipedia: norm

Definition 22.1 The *norm* of a vector $x \in \mathbb{R}^n$ is defined as:

$$\|x\| = \sqrt{\langle x, x \rangle}$$

We have the following properties of the norm:

- $\|x\| \geq 0, \forall x \in \mathbb{R}^n$.
- $\|x\| = 0$ if and only if $x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$
- $|\langle x, y \rangle| \leq \|x\| \|y\|$.

Although it's a little weird, we can use cosine in order to define angles.

Definition 22.2 The *angle* between two vectors $x, y \in \mathbb{R}^n$ is the unique $\alpha \in [0, \pi]$ such that $\cos(\alpha) = \frac{\langle x, y \rangle}{\|x\| \|y\|}$. We say that two vectors $x, y \in \mathbb{R}^n$ are *orthogonal* (or *perpendicular*), written $x \perp y$ if $\langle x, y \rangle = 0$. A vector x is called a *unit* vector (or *normalized vector*) if $\|x\| = 1$.

We can normalize any vector we want by dividing it by its length, so that its length is 1. In other words $y = \frac{x}{\|x\|}$ is always a unit vector (assuming $\|x\| \neq 1$).

Definition 22.3 A basis $B = \{u_1, \ldots, u_n\}$ is said to be *orthogonal* if $\langle u_i, u_j \rangle = 0$ for all $1 \leq i < j \leq n$. We say B is *orthonormal* is additionally every basis vector is a unit, *i.e.*, $\|u_k\| = 1$ for all $1 \leq k \leq n$.

23 Gram-Schmidt process

It turns out every basis can be used to construct an orthonormal basis. This process is called the *Gram-Schmidt process*

Wikipedia: Gram-Schmidt process

Gram-Schmidt process: Let $B = \{x_1, \ldots, x_n\}$ be a basis. We obtain an orthogonal basis $B' = \{y_1, \ldots, y_n\}$ and an orthonormal basis $B'' = \{z_1, \ldots, z_n\}$ in the following way:

(1) Set $y_1 = x_1$ and $z_1 = \frac{y_1}{\|y_1\|}$.

(2) Let $y_2 = x_2 + \lambda y_1$ such that y_2 is orthogonal to y_1.

$$\begin{aligned} 0 &= \langle y_2, y_1 \rangle \\ &= \langle x_2 + \lambda y_1, y_1 \rangle \\ &= \langle x_2, y_1 \rangle + \lambda \langle y_1, y_1 \rangle \end{aligned}$$

Therefore, we set $\lambda = -\frac{\langle x_2, y_1 \rangle}{\langle y_1, y_1 \rangle}$, implying:

$$y_2 = x_2 - \frac{\langle x_2, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 \qquad z_2 = \frac{y_2}{\|y_2\|}$$

(3) Let $y_3 = x_3 + \lambda_1 y_1 + \lambda_2 y_2$ and since we want an orthogonal basis, make y_3 orthogonal to y_2 and y_1.

$$\begin{aligned} 0 &= \langle y_3, y_1 \rangle \\ &= \langle x_3 + \lambda_1 y_1 + \lambda_2 y_2, y_1 \rangle \\ &= \langle x_3, y_1 \rangle + \lambda_1 \langle y_1, y_1 \rangle + \lambda_2 \langle y_2, y_1 \rangle \\ &= \langle x_3, y_1 \rangle + \lambda_1 \langle y_1, y_1 \rangle \\ &\Rightarrow \lambda_1 = -\frac{\langle x_3, y_1 \rangle}{\langle y_1, y_1 \rangle} \\ 0 &= \langle y_3, y_2 \rangle \\ &= \langle x_3 + \lambda_1 y_1 + \lambda_2 y_2, y_2 \rangle \\ &= \langle x_3, y_2 \rangle + \lambda_1 \langle y_1, y_2 \rangle + \lambda_2 \langle y_2, y_2 \rangle \\ &= \langle x_3, y_2 \rangle + \lambda_2 \langle y_2, y_2 \rangle \\ &\Rightarrow \lambda_2 = -\frac{\langle x_3, y_2 \rangle}{\langle y_2, y_2 \rangle} \end{aligned}$$

Therefore, we set

$$y_3 = x_3 - \frac{\langle x_3, y_1\rangle}{\langle y_1, y_1\rangle} y_1 - \frac{\langle x_3, y_2\rangle}{\langle y_2, y_2\rangle} y_2 \qquad z_3 = \frac{y_3}{\|y_3\|}$$

(4) Generally, we have:

$$y_k = x_k - \sum_{i=1}^{k-1} \frac{\langle x_k, y_i\rangle}{\langle y_i, y_i\rangle} y_i \qquad z_k = \frac{y_k}{\|y_k\|}$$

Let's see an example of this.

Example 23.1 Suppose we have the following basis. Using the Gram-Schmidt process, obtain an orthonormal basis.

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

We start with z_1

Now we work on z_2. First let's calculate y_2.

We finally find z_3.

24 Orthogonal complement

Given a linear subspace L of a vector space $\mathbb{R}^n$, we can construct a linear subspace whose basis vectors are all orthogonal to the basis of L.

Definition 24.1 Let L be a linear subspace of $\mathbb{R}^n$. The *orthogonal complement* of L is the linear subspace $L^\perp$ such that

Wikipedia: orthogonal complement

$$L^\perp - \{y \in \mathbb{R}^n \;\mid\; \langle x, y\rangle = 0, \forall x \in L\}$$

If $\{a_1, \ldots, a_n\}$ is a basis of L, then, by definition, $\langle x, a_i\rangle = 0$ for every vector $x \in L^\perp$. Therefore, the implicit equations of $L^\perp$ are given by:

$$\begin{cases} \langle x, a_1\rangle &= 0 \\ \langle x, a_2\rangle &= 0 \\ \vdots \\ \langle x, a_n\rangle &= 0 \end{cases}$$

Example 24.2 Calculate the orthogonal complement of the subspace L given the following basis of L:

$$B_L = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

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Bibliography

These notes were based off the following.

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