Calculus I

Class Notes - Student Version

University of Loyola

Bachelors in Applied Mathematics

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Contents

Welcome

Welcome to Calculus I! This semester we'll be going through a ton of various stuff in order to get you better equipped with the tools of calculus. These notes are designed to better help you learn the material and guide you through the course.

How to use these notes: These notes are different from most math class notes. They have holes and gaps in them; they are missing information! These are the notes that I will be using in my class lectures (literally) and so it should help you follow along. The purpose of these notes is so that you don't have to write everything down; you only need to fill in the gaps giving you more time to listen to lectures and better understand the material. If you need to add extra notes, there is tons of room in the margins, use them! These are YOUR notes. I've also added links to Wikipedia in case you want another reference to the terms and definitions we go over. If you feel using these notes would hinder your learning experience, then please use another system. The point is to learn; so use whatever works best for you.

Before starting a lecture I'd recommend quickly glancing over the notes (5-10 minutes) in order to get a brief idea of what we'll be covering in class. Remember to review these notes periodically so that you don't forget terms and examples.

Good luck, I believe in you!

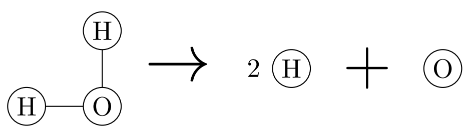
-Aram

Chapter 1

Elementary functions

1 Functions

A function is nothing more than a rule where you give it something and get back something else. As an example:



Definition 1.1 More explicitly, a *function* f is a rule that assigns to each x in a set D to *exactly* one element ________, in a set E. This is normally denoted ____________________ where D is called the *domain* and E is called the *codomain*. For $x \in D$, then $f(x) \in E$ is called the *image* of x and x is called a *preimage* of $f(x)$. The *range* of f is the set of all possible values of $f(x)$ for every x in D.

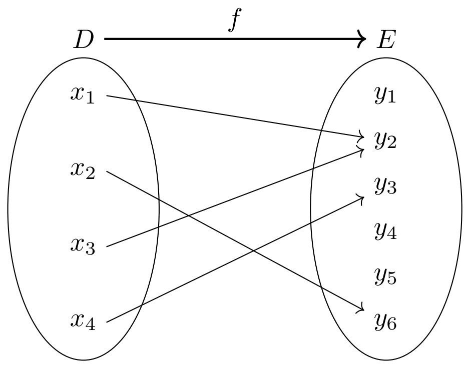
Wikipedia: function

Wikipedia: domain

Wikipedia: image

Problem 1.2 Create your own function that does not use any numbers nor atoms! When finished, compare with your neighbor.

Example 1.3 Let $D = \{x_1, x_2, x_3, x_4\}$ and $E = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. We can view the function in the following way:



- Domain: ____
- Codomain: ____
- Range: ____________________
- Image: What's the image of x_2? ____;
- Preimage: What's the preimage of y_2? _______________.

2 Simple functions

Let's look at some examples.

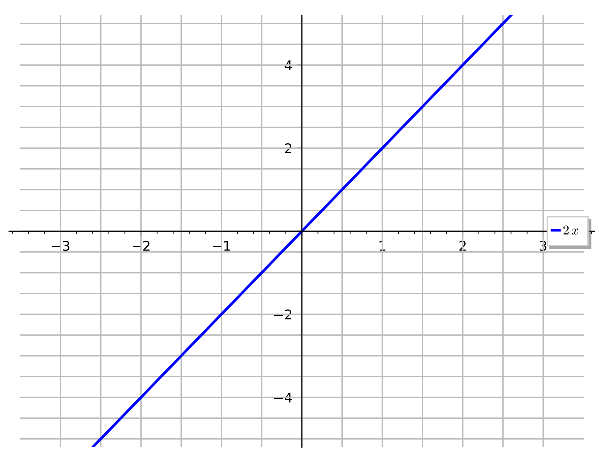
2.1 Linear function

Wikipedia: linear function

Definition 2.1 A *linear function* is a function $f : \mathbb{R} \to \mathbb{R}$ whose graph is a line (*e.g.* functions of the form $f(x) = mx + b$ where $m, b \in \mathbb{R}$).

Example 2.2 Let f be the linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x$.

Domain: ___ Codomain: ___ Range: ___



2.2 Polynomial functions

Definition 2.3 A *polynomial* is a function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$$

where $a_0, \ldots, a_n$ are known as the *coefficients* and, if $a_n \neq 0$, then f is said to have *degree n* .

Wikipedia: coefficients
Wikipedia: degree n

Some polynomials have special names depending on the degrees n:

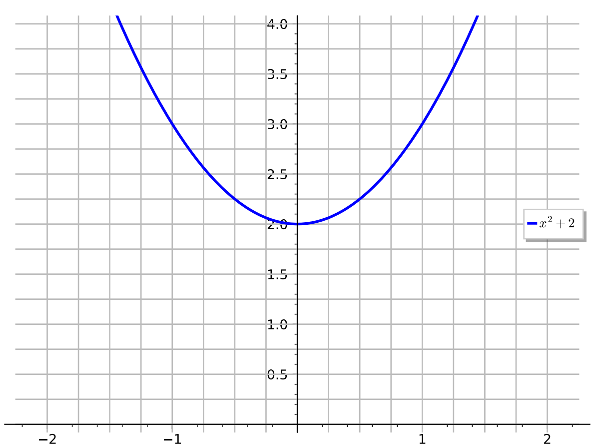
- If $n = 1$ then it's a linear function.
- If $n = 2$ then it's a *quadratic function* .
- If $n = 3$ then it's a *cubic function* .

Wikipedia: quadratic function
Wikipedia: cubic function

There's more (quartic, quintic, sextic, septic, octic, nonic, decic, etc.), but we'll stop here.

Example 2.4 Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial of degree 2 such that $f(x) = x^2 + 2$.

Domain: ___ Codomain: ___ Range: _______



2.3 Rational functions

Definition 2.5 A *rational function* f is a ratio of two polynomials $P(x)$ and $Q(x)$:

Wikipedia: rational function

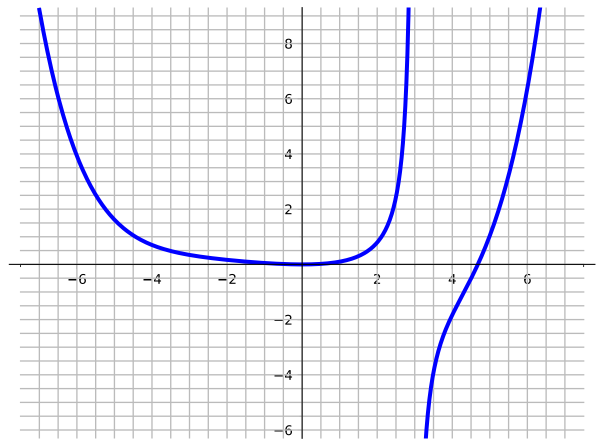
$$f(x) = \frac{P(x)}{Q(x)}.$$

Example 2.6 Let f be the rational function

$$f(x) = \frac{x^7 - x^5 - 7x^4 + x^3 - 2000x^2 + 67}{10000(x-3)}$$

Domain: ___________ Codomain: ___

Why am I not asking for the codomain? ____________________________



Example 2.7 Are the functions $f(x) = x$ and $g(x) = \frac{x^2}{x}$ the same function? ____________________________

2.4 Piecewise functions

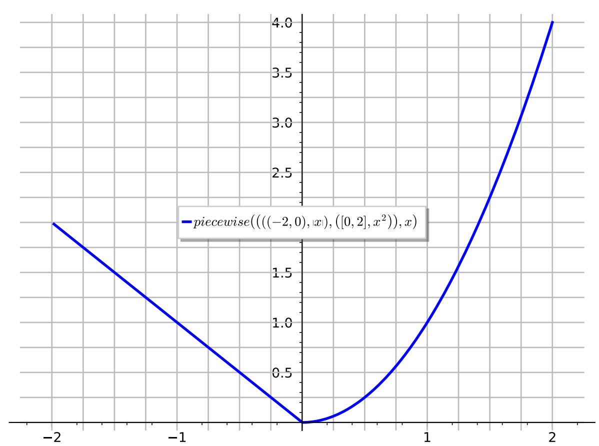
Wikipedia: piecewise function

Definition 2.8 A *piecewise function* is a function which has different formulas in different parts of their domains. This is best seen through an example.

Example 2.9 Let f be the piecewise defined function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) =$$

Domain: ___ Codomain: ___ Range: ______

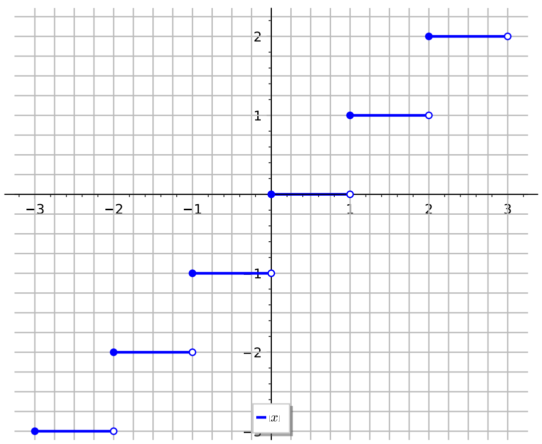


2.5 Floor and ceiling functions

Definition 2.10 Let f be the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \lfloor x \rfloor$. Here $\lfloor x \rfloor$ means that we round down to the nearest whole number. So for example $\lfloor \pi \rfloor = 3$, $\lfloor 2.4 \rfloor = 2$, $\lfloor 1 \rfloor = 1$ and $\lfloor -2.2 \rfloor = -3$. This function is known as the *floor function* . There is a similar function $g(x) = \lceil x \rceil$ which rounds up instead which is called the *ceiling function*.

Wikipedia: floor function

Domain: ___ Codomain: ___ Range: ___



3 Properties

3.1 Injective, surjective and bijective functions

Definition 3.1 Let $f : D \to E$ be a function.

We say f is *injective* if each element in the range has a unique preimage. In other words, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Wikipedia: injective

We say f is *surjective* if the range is the entire codomain.

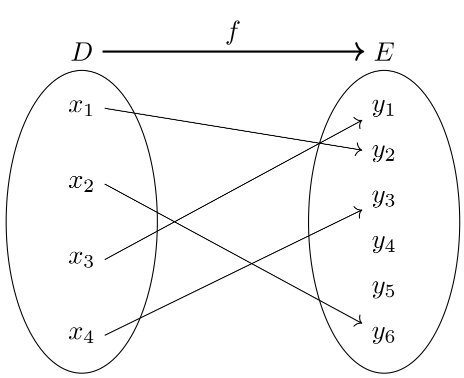
Wikipedia: surjective

Finally, we say that f is *bijective* if it is both injective and surjective.

Wikipedia: bijective

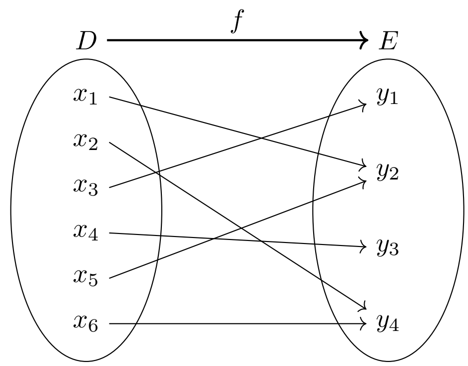
Let's see some examples:

Example 3.2



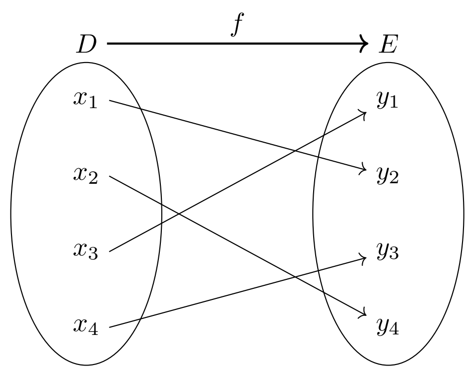
Injective: _______ Surjective: ______ Bijective: ______

Example 3.3



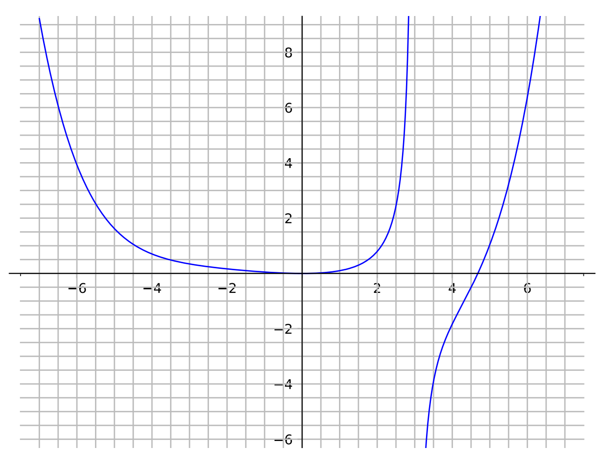
Injective: ______ Surjective: _______ Bijective: ______

Example 3.4



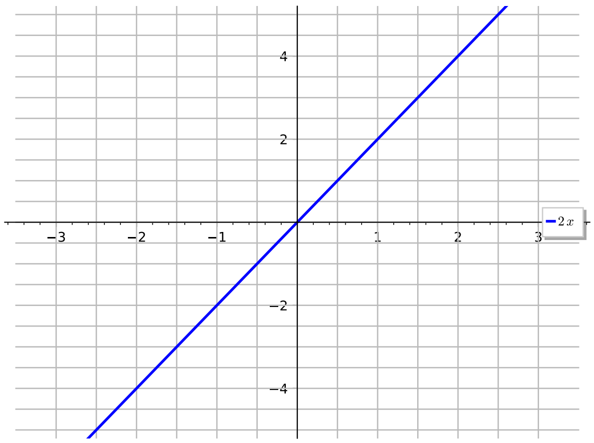
Injective: _______ Surjective: _______ Bijective: _______

Example 3.5



Injective: ______ Surjective: _______ Bijective: ______

Example 3.6



Injective: _______ Surjective: _______ Bijective: _______

Proposition 3.7 *A function is one-to-one if and only if* ___________
____
____

3.2 Even and odd functions

Definition 3.8 An *even* function is a function f that satisfies $f(-x) = f(x)$. An *odd* function is a function f that satisfies $f(-x) = -f(x)$.

Wikipedia: even

Example 3.9 Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) =$ _____________. Is it even or odd? _______

Example 3.10 Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \sin(x)$ and $g : \mathbb{R} \to \mathbb{R}$ be the function $g(x) = \cos(x)$. Which is even and which is odd?

Even: ______ Odd: _____

Problem 3.11 With a person next to you: Give an example of a function that is neither even nor odd:

Problem 3.12 With a person next to you: Give an example of a function that is both even and odd:

3.3 Monotone functions

Before we begin, recall that $[a, b]$ is a *closed interval* of $\mathbb{R}$ and (a, b) is an *open interval* of $\mathbb{R}$.

Wikipedia: closed interval

Definition 3.13 A function $f : D \to E$ is called *increasing* on an interval I if:

$$f(x) \leq f(y) \text{ for every } x \leq y \text{ in } I.$$

A function $f : D \to E$ is called *decreasing* on an interval I if:

$$f(x) \geq f(y) \text{ for every } x \leq y \text{ in } I.$$

Wikipedia: monotonic

A function $f : D \to E$ is *monotonic* if it is either increasing or decreasing.

3.4 Operations

Definition 3.14 Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be two functions. Then,

- $f + g : D \cap E \to \mathbb{R}$, such that $(f + g)(x) = f(x) + g(x)$
- $f - g : D \cap E \to \mathbb{R}$, such that $(f - g)(x) = f(x) - g(x)$
- $fg : D \cap E \to \mathbb{R}$, such that $(fg)(x) = f(x) \cdot g(x)$
- $\frac{f}{g} : (D \cap E - \{x \in E \mid g(x) = 0\}) \to \mathbb{R}$, such that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Example 3.15 Let $f(x) = x^2$ and $g(x) = x - 1$.

$(f + g)(x) = \underline{\qquad\qquad\qquad}$ $(f - g)(x) = \underline{\qquad\qquad\qquad}$

$(fg)(x) = \underline{\qquad\qquad}$ $\dfrac{f}{g}(x) = \underline{\qquad\qquad}$

Wikipedia: composition

Definition 3.16 The *composition* of two functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$ where

$$g \circ f(x) = \underline{\qquad\qquad}$$

The order matters for composition, just like subtraction and division!

Example 3.17 Let $f(x) = x^2$ and $g(x) = x - 1$.

$(f \circ g)(x) = \underline{\qquad\qquad}$ $(g \circ f)(x) = \underline{\qquad\qquad}$.

Problem 3.18 With the person next to you: Let $f(x) = x + 3$ and $g(x) = \frac{1}{x^2+1}$.

$(f \circ g)(x) = \underline{\qquad\qquad\qquad}$ $(g \circ f)(x) = \underline{\qquad\qquad\qquad}$.

3.5 Inverse

A function allows us to map some number a to some number b using a formula, but what if we want to go backwards? How do we find a function that allows us to go from b to a, and is this even possible?

Example 3.19 Let's try an easy example. Let f be the function $f(x) = x + 1$. To go backwards we would need to use the formula _________.

But this doesn't always work!

Example 3.20 Let f be the function $f(x) = x^2$. If I know $f(x) = 4$, what is x? ____

What rule do we need to make sure that this *does* always work? The function needs to be ______________.

Definition 3.21 Let $f : A \to B$ be an ______________ function. Then its *inverse function* is the ______________ function $f^{-1} : B \to A$ and is defined by

Wikipedia: inverse function

Example 3.22 Taking our example from before for $f(x) = x^2$. If instead of defining $f : \mathbb{R} \to \mathbb{R}$ we define it as $f : \mathbb{R}^+ \to \mathbb{R}^+$, then it does have an inverse. Its inverse function is ______.

How do we find the inverse of a function?

(1) ________________________

(2) ____

(3) ____________________

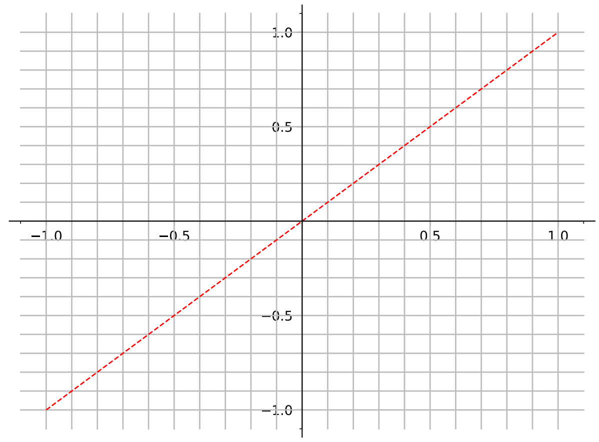
Example 3.23 If $f(x) = x^2$ then

(1) _________________________.

(2) ____________.

(3) _____________.

Graphically, we can think of this as reflecting over the $y = x$ line.



Inverse and reciprocal are not always the same!

$$f^{-1}(x) \neq (f(x))^{-1}$$

Problem 3.24 With the person next to you: Let $f(x) = x^3 + 1$. What is its inverse function?

$$f^{-1}(x) = \underline{\hspace{3cm}}$$

What is its reciprocal function?

$$(f(x))^{-1} = \underline{\hspace{3cm}}$$

4 More complex functions

4.1 Exponential and logarithmic functions

Exponential

Wikipedia: exponential function

Definition 4.1 An *exponential function* is a function of the form

$$f(x) = \underline{\hspace{1cm}}$$

where b is a positive constant (*i.e.*, $b \in \mathbb{R}^+$).

How do we calculate b^x for every $x \in \mathbb{R}$?

x	b^x	Example
Integer ($\mathbb{Z}$)	b^x	
Rational ($\mathbb{Q}$)	$b^{p/q} = \sqrt[q]{b^p}$	
Irrational ($\mathbb{R}\backslash\mathbb{Q}$)	Estimate using graph	

Example 4.2 Let $f(x) = 2^x$ and let $g(x) = x^2$.

$$(f \circ g)(x) = \underline{\hspace{2cm}} \qquad (g \circ f)(x) = \underline{\hspace{2cm}}$$

Are these the same function? (*i.e.*, is $f \circ g = g \circ f$) ____ .

Logarithm

For exponential, we let its inverse be called the *logarithm*. More precisely, we have the following definition.

Definition 4.3 For $f(x) = b^x$ (where $b > 0$ and $b \neq 1$), the exponential passes the horizontal line test. The *logarithm* function with base b, denoted _______, is the inverse of $f(x)$, *i.e.*, $f(x) = b^x$ implies $f^{-1}(x) =$ ____________.

Wikipedia: logarithm

Example 4.4 Let $f(x) = 10^x$.

Some nice properties of logarithms:

- $\log_b(xy) =$ ______________________________
- $\log_b(\frac{x}{y}) =$ ______________________________
- $\log_b(x^y) =$ _______________

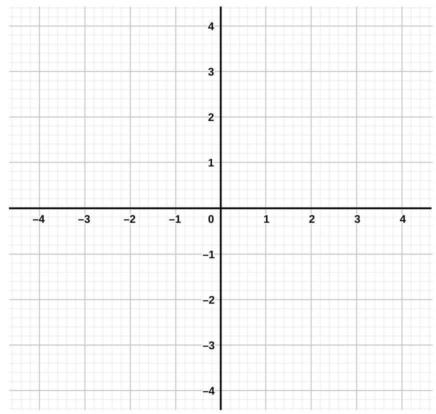
When $b =$ __, the logarithm is called the *natural logarithm* and is denoted _________.

Some books will write $\log(x)$ (no subscript) to mean either $\log_{10}(x)$ or $\log_e(x) = \ln(x)$. Always double check with the book to ensure you know which version it is.

4.2 Trigonometric functions

Trig functions

Aram's favorite way to view trig functions:



This helps us calculate all the other trig functions as well:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \ \csc(\theta) = \frac{1}{\sin(\theta)}, \ \sec(\theta) = \frac{1}{\cos(\theta)}, \ \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Although we had the radius be equal to 1, we can alter the radius! Then we need a new method to solve for our trig functions. This is where an acronym comes in: ________________. This then gives us the following ways to calculate all the trig functions.

There are a few trig functions that are worthwhile to know how to quickly calculate.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$--$

Here are some trig identities which are good to know:

(1) $\sin^2(x) + \cos^2(x) = 1$

(2) $\tan^2(x) + 1 = \sec^2(x)$

(3) $1 + \cot^2(x) = \csc^2(x)$

(4) $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

(5) $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

(6) $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$

(7) $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$

(8) $\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$

(9) $\sin(x) \pm \sin(y) = 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right)$

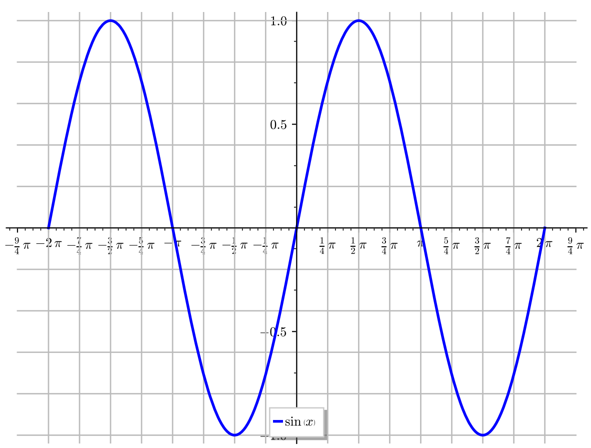
(10) $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

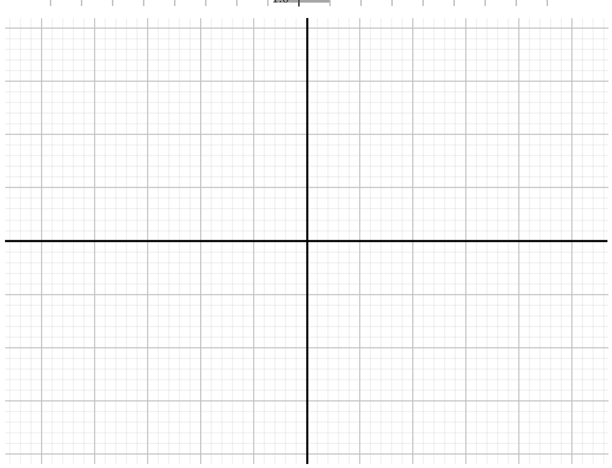
(11) $\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Inverse trig functions

Let's start with $\sin(x)$. How do we figure out its inverse? Recall that you can just reflect over the __________ line to find the inverse. So let's do that.

Problem 4.5 Reflect $f(x) = \sin(x)$ over the __________ line.





Turn to your neighbor and compare your drawing. Does this reflection give us an inverse function? _____. How do we get an inverse function? ________________________________

The inverse function for sine is known as the *arcsine* function and is the function:

Example 4.6 What is $\arcsin\left(\frac{\sqrt{3}}{2}\right)$? ______

We can similarly do the same thing for cosine! The inverse function for cosine is known as the *arccosine* function.

Example 4.7 What is $\arccos(\frac{1}{2})$? ______

Problem 4.8 With a partner, solve the following:

(1) $\arcsin(\frac{1}{2}) =$ ______

(2) $\arccos(\frac{\sqrt{3}}{2}) =$ ______

(3) $\arcsin(1) =$ ___

(4) $\arccos(1) =$ __

The inverse function for tangent is known as the *arctangent* function.

There are 3 other trigonometric functions:

$$\csc = \frac{1}{\sin} \qquad \sec = \frac{1}{\cos} \qquad \text{cotan} = \frac{1}{\tan}$$

Let's put all of this into a nice table:

Function	Domain	Range
$\arcsin(x)$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\arccos(x)$	$[-1,1]$	$[0,\pi]$
$\arctan(x)$	$\mathbb{R}$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\operatorname{arccsc}(x)$	$\lvert x\rvert \geq 1$	$\left[-\frac{\pi}{2},0\right) \cup \left[0,\frac{\pi}{2}\right]$
$\operatorname{arcsec}(x)$	$\lvert x\rvert \geq 1$	$\left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$
$\operatorname{arccot}(x)$	$\mathbb{R}$	$[0,\pi]$

Problem 4.9 With a partner, solve the following:

(1) $\arctan(1) =$ ______.

(2) $\arctan(0) =$ __.

(3) $\operatorname{arccsc}(\sqrt{2}) =$ ______.

Hyperbolic

We won't use these too much in this course, but note that there are trig functions known as *hyperbolic trig functions*. We put them here for future use:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Chapter 2

Sequences and series

5 Sequences

A lot of times in math, we work with a bunch of numbers. Usually these numbers come in some sort of order such as

$$1,\ 2,\ 3,\ 4,\ \ldots$$
$$1,\ 1,\ 1,\ 1,\ \ldots$$
$$1,\ -4,\ 9,\ -16,\ \ldots$$

We can use math in order to more explicitly define these sequences of numbers.

Definition 5.1 A *real number sequence* a is a map which assigns to each $n \in \mathbb{N}$ a real number:

Wikipedia: real number sequence

$$a : \mathbb{N} \to \mathbb{R}.$$

We frequently use a_n to denote ________ and write a as a sequence of numbers a_1, a_2, ... We also will write $\{a_n\}$ to denote the sequence itself, *e.g.* ______________________________. The n in a_n is commonly referred to as the *index*.

Example 5.2 (1) $1, 1, 1, 1, 1, \ldots = \{1\}$

(2) $1, 2, 3, 4, 5, \ldots = \{n\}$

(3) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots =$ ____________

(4) $1, 2, 6, 24, 120, \ldots =$ ________

(5) $1, 1, 2, 3, 5, 8, 13, 21, \ldots = \left\{\frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}\right\}$ where $\varphi = 1.61803398874989\ldots$.

(6) $1, 1, 2, 5, 14, 42, \ldots = \left\{\frac{1}{n+1}\binom{2n}{n}\right\}$

(7) Make your own: ____________________________

Example 5.3 The series

$$\{a_n\} = 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots$$

has elements:

This sequence is commonly known as the Fibonacci sequence and is best defined recursively.

Wikipedia: Fibonacci sequence

5.1 Recursive sequences

The previous examples defined a sequence using a rule based on the index of the elements (*e.g.* n, $\frac{1}{n}$, etc.) or was defined directly using a sequence of numbers.

Another way to define a sequence of numbers is to use previous terms in order to define the next terms. This type of sequence is called a *recursive sequence* .

Wikipedia: recursive sequence

Example 5.4

(1) $1, 1, 1, 1, 1, \ldots$: $a_1 = 1$ and $a_n = a_{n-1}$.

(2) $1, 2, 3, 4, 5, \ldots$: $a_1 = 1$ and $a_n = a_{n-1} + 1$.

(3) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$: $a_1 = \frac{1}{2}$ and $a_n = \frac{n-1}{n} a_{n-1}$.

(4) $1, 2, 6, 24, 120, \ldots$: $a_1 =$ __ and $a_n =$ __________.

(5) $1, 1, 2, 3, 5, 8, 13, 21, \ldots$: $a_1 =$ __ and $a_n =$ ______________________.

(6) $1, 1, 2, 5, 14, 42, \ldots$: $a_1 = 1$ and $a_n = \frac{2(2n-1)}{n+1} a_{n-1}$.

(7) Make your own: ____________________________

5.2 Graphical representation

The graphical representation of a sequence can help in analysing its behaviour, allowing us to see if the sequence goes towards a particular number or towards infinity.

Example 5.5 The map $c_n = \frac{1}{n}$ gives the sequence:

$$\{c_n\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

which can be represented by the following graph:

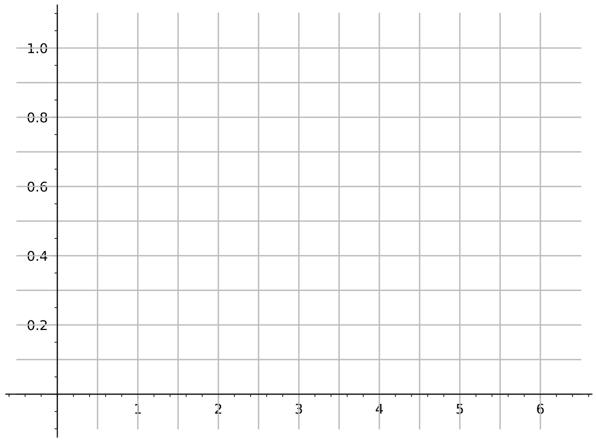


Figure 2.1: Sequence - graphical

or by the following straight line:

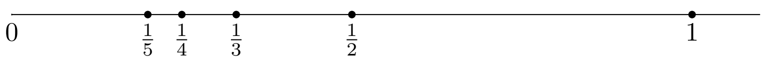


Figure 2.2: Sequence - straight line

Usually we will use the graph as it also allows us to know how the sequence is changing.

5.3 Bounds

Definition 5.6 A sequence $\{a_n\}$ is *bounded above* if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. In such a case, we say that M is an *upper bound* of $\{a_n\}$.

Wikipedia: upper bound

Example 5.7 The sequence

$$\{b_n\} = -1, 1, -1, 1, -1, 1, \ldots$$

has the numbers __, __, __, __, ... as upper bounds and thus is bounded above.

Definition 5.8 A sequence $\{a_n\}$ is *bounded below* if there exists $m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \in \mathbb{N}$. In such a case, we say that M is a *lower bound* of $\{a_n\}$.

Example 5.9 The sequence

$$\{a_n\} = \left\{\frac{n+1}{n}\right\} = 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$$

has the numbers ______, ______, __, __, ... as lower bounds and thus is bounded below.

Definition 5.10 A sequence $\{a_n\}$ is *bounded* if there exists $m, M \in \mathbb{R}$

such that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$. In other words, $\{a_n\}$ has a lower bound and an upper bound.

Example 5.11 The sequence

$$\{a_n\} = \left\{\frac{4 - 3n}{n^2}\right\}$$

with the following graphical representation

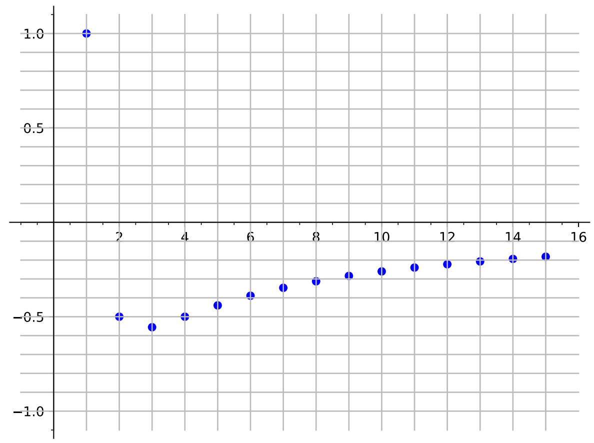


Figure 2.3: Bounded sequence

has -1 as a lower bound and 1 as an upper bound, and therefore is a bounded sequence.

To show (mathematically) that a sequence is bounded, we must show that there exists m, $M \in \mathbb{R}$ such that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$.

We will consider three different methods for showing this:

- Algebra
- Known bounds
- Induction

Algebra

Using the equation that defines the sequence, and known algebraic definitions, it's possible to show the existence of bounds of a sequence.

Example 5.12 Show that the sequence $\{a_n\} = \left\{\frac{n}{n+1}\right\}$ has an upper bound.

In this case, we start by comparing the numerator and denominator with the quotient. We know that $n < n + 1$ for every $n \in \mathbb{N}$. Therefore, if we divide both sides of the inequality by $n + 1$ we have $\frac{n}{n+1} < 1$ for all $n \in \mathbb{N}$ since $n \geq 0$. In other words:

$$a_n < 1 \qquad \forall n \in \mathbb{N}$$

implying that 1 is an upper bound.

Example 5.13 Show that the sequence $\{b_n\} = \left\{3 - \frac{5}{n}\right\}$ is bounded.

We know that for all $n \in \mathbb{N}$ we have that __ $\leq \frac{1}{n} \leq$ __. If we multiply both sides by -5 and add 3, we get that

$$\forall n \in \mathbb{N}$$

implying that _____ is a lower bound and __ is an upper bound. Therefore, $\{b_n\}$ is bounded.

Known bounds

Many elementary functions have known bounds that we can use to deduce the bounds of similar functions. This method is useful when the sequence is defined by trigonometric functions, exponentials, logarithms, etc.

Example 5.14 Suppose $\{a_n\} = \left\{\sin\left(\frac{n^2+1}{3n^2+4n+5}\right)\right\}$. By definition, we know that the function $\sin(x)$ is bounded by -1 and 1. We also know that alterations to the variable x change the frequency of the function, but not the amplitude. In other words, the function continues being bounded between -1 and 1.

Example 5.15 Show that the sequence $\{c_n\} = \{5 + 4\cos(n)\}$ is bounded.

We know that ____________________________ for all $n \in \mathbb{N}$. Therefore, multiplying both sides by 4 and adding 5 to each side gives:

and thus our sequence is bounded.

Induction

Induction is a rigorous way to show that a proposition is true for all natural numbers. Proving something by induction involves three steps:

Wikipedia: Induction

Base: Show that the proposition is true for $n = 1$.

Assumption: Assume that the proposition is true for $n = k$.

Inductive step: Show that the proposition is true for $n = k + 1$.

In the base step note that we can technically start from any number n if our proposition asks us to prove something for all natural numbers greater than n.

Example 5.16 Show that the (recursive) sequence $\{a_{n+1}\}$ defined by $a_1 = 0$ and $a_{n+1} = a_n^2 + \frac{1}{4}$ is bounded above by $\frac{1}{2}$.

We prove this proposition using induction.

Base: When $n = 1$, then by definition $a_1 = 0$. Therefore, since $a_1 = 0 < \frac{1}{2}$, our proposition is true for $n = 1$.

Assumption: We assume that our proposition is true for $n = k$. In other words that $a_k \leq \frac{1}{2}$ for $k \in \mathbb{N}$.

Inductive step: We need to show that $a_{k+1} \leq \frac{1}{2}$. Recall that $a_{k+1} = a_k^2 + \frac{1}{4}$. Furthermore, since $a_k \leq \frac{1}{2}$ we know $a_k^2 \leq \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Putting these together we have:

$$a_{k+1} = a_k^2 + \frac{1}{4} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Example 5.17 Prove that the sequence defined by $\{a_n\} = \left\{\frac{4-3n}{n^2}\right\}$ is bounded above by 1.

As before, we prove this proposition by induction.

Base: When ____

____

____

Assumption: Suppose that the proposition is true for __________. In other words, ______________________________________

____

Inductive step: We need to show that the proposition is true for ________________. By definition, we know that $a_{k+1} =$ ______________. We have:

Exercise 5.18 Prove that the (recursive) sequence $\{x_n\}$ defined by $x_1 = 1$ and $x_{n+1} = \sqrt{1 + 2x_n}$ is bounded above by 4 for all $n \in \mathbb{N}$.

5.4 Infimum and Supremum

The *infimum* of a sequence is the ______________________________ ______, or in other words, it is the maximal element in the set of all lower bounds. A lower bound m_0 is the infimum of a sequence $\{a_n\}$ if

$$m_0 \geq m \qquad \text{for all lower bounds } m \text{ of } \{a_n\}.$$

Wikipedia: infimum

We denote the infimum as either m_0 or $\inf(a_n)$.

The *supremum* of a sequence is the __________________________ ______, or in other words, it is the minimal element in the set of all upper bounds. An upper bound M_0 is the supremum of a sequence $\{a_n\}$ if

$$M_0 \leq M \qquad \text{for all upper bounds } M \text{ of } \{a_n\}.$$

We denote the supremum as either M_0 or $\sup(a_n)$.

Theorem 5.19 (Uniqueness of the supremum) *If a sequence has a supremum, it is unique.*

Proof. Let $\{a_n\}$ be a sequence of real numbers. Suppose that the sequence has two supremums b and c.

Since c is a supremum, it is an upper bound of $\{a_n\}$. If b is a supremum then by definition $b \leq c$ as c is an upper bound of $\{a_n\}$. Similarly, if b is a supremum, it is an upper bound of $\{a_n\}$ and thus c being a supremum implies $c \leq b$.

If $b \leq c$ and $c \leq b$ then $b = c$. Therefore, if a supremum exists, then it is unique. □

5.5 Monotone sequences

Definition 5.20 A sequence $\{a_n\}$ is *increasing* if $a_n \leq a_{n+1}$ for all $N \in \mathbb{N}$. If $a_n < a_{n+1}$ then we say the sequence is *strictly increasing*.

Example 5.21 The sequence

$$\{a_n\} =$$

has elements such that:

therefore $\{a_n\}$ is a ________________________________ sequence.

Definition 5.22 A sequence $\{a_n\}$ is *decreasing* if $a_n \geq a_{n+1}$ for all $N \in \mathbb{N}$. If $a_n > a_{n+1}$ then we say the sequence is *strictly decreasing*.

Example 5.23 The sequence

$$\{a_n\} =$$

has elements such that:

therefore $\{a_n\}$ is a ________________________________ sequence.

Definition 5.24 A sequence $\{a_n\}$ is *monotone* if it is either __________ ____________________________________ sequence.

There are three main ways to show that a sequence is monotone.

- Difference
- Logic
- Quotient

Difference

In this method, we consider the difference $a_{n+1} - a_n$. If the difference is positive for all n, then the sequence is increasing. If the difference is negative for all n, then the sequence is decreasing.

Example 5.25 Given the sequence $\{a_n\} = \{n^2\}$.

We have for arbitrary $n \in \mathbb{N}$

Therefore, $\{a_n\}$ is an ________________ sequence.

Logic

In this method, we use logic. To prove that a sequence is increasing, we show that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, or, more specifically, that $m \geq n$ implies $a_m \geq a_n$. On the other hand, to show that a sequence is decreasing, we show that $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$, or, more specifically, that $m \geq n$ implies $a_m \leq a_n$.

Example 5.26 Given the sequence $\{a_n\} = \{r^n\}$ where, $r > 1$. We have,

since $r > 1$. Therefore, $\{a_n\}$ is an ________________ sequence.

Quotient

In this method, we consider the ratio $\frac{a_{n+1}}{a_n}$. If the ratio is greater than 1 for all $n \in \mathbb{N}$, and a_n is positive, then $\{a_n\}$ is an increasing sequence. If the ratio is less than 1 for all $n \in \mathbb{N}$, and a_n is positive, then $\{a_n\}$ is a decreasing sequence.

Example 5.27 Given the sequence $\{a_n\} = \{\frac{n}{n+1}\}$. We have that $a_n > 0$ for all $n \in \mathbb{N}$ and

Therefore, $\{a_n\}$ is an ________________ sequence.

Problem 5.28 Given the sequence $\{b_n\} = \{n^2\}$. With a partner show that it is increasing.

Exercise 5.29 Given the sequence $\{c_n\} = \{n!\}$. Show that it is increasing.

Therefore $\{c_n\}$ is an increasing sequence.

5.6 Limits of sequences

Sometimes when we go further and further down the sequence, the sequence seems to approach a number. For example, if we look at the sequence:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Let's look at what this looks like in graph form

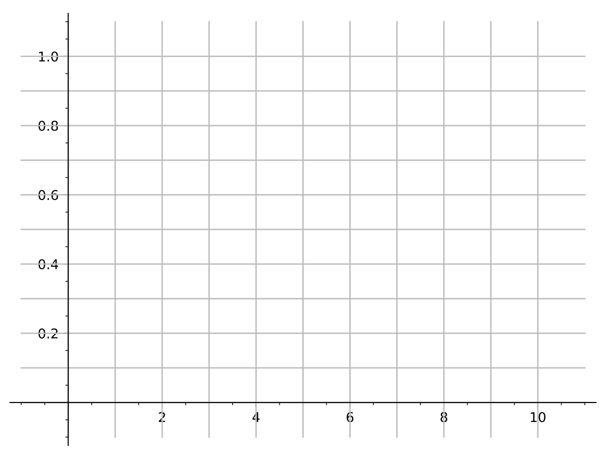


Figure 2.4: Limit of a sequence

It looks like this sequence is going towards: __.

In order to encapsulate this idea, in mathematics we have the concept of "convergence". It's a little technical, but all it's doing is writing explicitly what we mean by a sequence approaching some number.

Definition 5.30 A sequence $\{a_n\}$ is *convergent* if there exists $L \in \mathbb{R}$ such that for all $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ there exists a number $N \in \mathbb{N}$

such that $|a_n - L| < \epsilon$. In shorthand, we have:

We call L the *limit* of the sequence and write:

$$\lim_{n\to\infty} a_n = L$$

The definition of a limit is fundamental in the proof of many theorems, but it's also useful for finding and verifying limits.

As it's confusing, we first give a general outline of how this proof method works.

Analysis: (1) ____

(2) ____

Proof: (1) ____

(2) ____

(3) ____

Finish Finish up the proof.

Let's see how to do this using an example. We show the limit in 3 steps.

Example 5.31 Show that $\lim_{n\to\infty}\left(4+\frac{2}{n^2}\right)=4$.

Analysis: We start with what we're trying to prove:

So this $\sqrt{\frac{2}{\varepsilon}}$ is the N we want to use in the next part.

Proof: We now use our analysis to construct a proof.

Let $\varepsilon > 0$ and let $N = \sqrt{\frac{2}{\epsilon}}$. Then for $n > N$ we have:

Finish Therefore, our series is convergent with limit $L = 4$.

Definition 5.32 If a sequence $\{a_n\}$ a sequence does *not* have a limit, then it is called *divergent*

Definition 5.33 If a sequence is divergent and oscillates between two points, it's sometimes called an *oscillating sequence* .

Wikipedia: oscillating sequence

Example 5.34 Take for example the following oscillating sequence:

$$\{a_n\} = 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

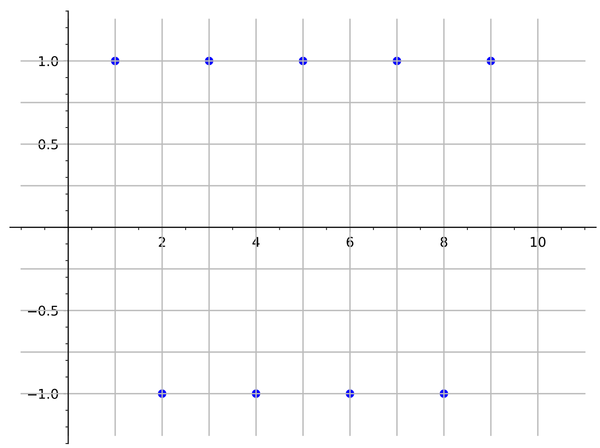


Figure 2.5: Oscillating sequence

Exercise 5.35 Can you prove why the previous sequence is divergent?

If we don't know what the limit of something is and need to show it's convergent/divergent, we can break the rules of math to help us. The rule of math we break is that we assume ∞ to be a number, and we plug it in. We then recall the following two properties to help us:

$$\lim_{n\to\infty} n = \underline{\qquad}$$
$$\lim_{n\to\infty} \frac{1}{n} = \underline{\quad}$$

Example 5.36 Determine if the following sequence is convergent, divergent and/or oscillating:

$$\{a_n\} = \left\{\frac{1}{2n} + \frac{1}{3n^2} + 1\right\}$$

We start by breaking the rules of math:

So we guess that:

$$\lim_{n\to\infty} \frac{1}{2n} + \frac{1}{3n^2} + 1 = 1$$

Example 5.37 Determine if the following sequence is convergent, divergent and/or oscillating:

$$b_n = n^{10} - n^8 - n^6$$

We start by breaking the rules of math:

So we guess that our series is divergent.

Note that this does *not* give us the limit and is *not* a proof. It's just to help us guess what the limit should be.

Problem 5.38 Try and guess what the limit of the following sequence is.

$$\{c_n\} = \left\{\frac{5n+3}{n+4} + \frac{3n^2-7}{n^2+8}\right\}$$

Guess: __.

Problem 5.39 Vote: Is the following sequence convergent, divergent or oscillating?

$$\{d_n\} = \{\cos(n)\}.$$

Problem 5.40 Try and guess what the limit of the following sequence is.

$$\{e_n\} = \{e^n\}$$

Guess: ________________.

Theorem 5.41 (Uniqueness of the limit) *Given a convergent sequence* $\{a_n\}$ *and two numbers* $b, c \in \mathbb{R}$ *such that* $\lim_{n\to\infty} a_n = b$ *and* $\lim_{n\to\infty} a_n = c$*, then* ________.

Proof. Suppose there are $b, c \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = b \text{ and } \lim_{n\to\infty} a_n = c$$

but that $b \neq c$. Furthermore, let $\varepsilon = \frac{1}{2}|b-c|$.

By the definition of the limit we have:

$$|a_n - b| < \varepsilon \qquad \forall n \geq N_b$$

$$|a_n - c| < \varepsilon \qquad \forall n \geq N_c$$

To ensure that the two inequalities hold, we let $n' > \max\{N_b, N_c\}$. In other words,

$$|a_{n'} - b| < \varepsilon \text{ and } |a_{n'} - c| < \varepsilon.$$

Then,

$$\begin{aligned}
|b-c| &= |b - a_{n'} + a_{n'} - c| \\
&\leq |b - a_{n'}| + |a_{n'} - c| && \text{(by triangle inequality)} \\
&< \varepsilon + \varepsilon \\
&= 2\varepsilon \\
&= |b-c|
\end{aligned}$$

But this is a contradiction, and so $b = c$. □

Theorem 5.42 (Monotone convergence theorem) *If the sequence $\{a_n\}$ is monotone and bounded, then it is* __________________.

Proof. Let $\{a_n\}$ be a sequence that is increasing and bounded above (the case for decreasing and bounded below is similar). The supremum of the sequence, S, has two properties:

- $a_n \leq S \quad \forall n \in \mathbb{N}$.
- For $\varepsilon > 0$, $S - \varepsilon$ isn't an upper bound (since it's smaller than S). Therefore, $\exists N \in \mathbb{N}$ such that $S - \varepsilon \leq a_N$.

The sequence is increasing, therefore for every $n \geq N$:

$$S - \varepsilon \leq a_N \leq a_n.$$

Using the first property of the supremum:

$$S - \varepsilon \leq a_n \leq S < S + \varepsilon$$

Then,

$$|a_n - S| < \epsilon.$$

But this is just the definition of the limit! Therefore, $\lim_{n\to\infty} a_n = S$. □

Not every convergent series is monotone and bounded.

Exercise 5.43 At home, find an example of a convergent series that is not monotone and bounded.

Corollary 5.44 *Let $\{a_n\}$ be a monotone bounded sequence.*

- *If it is increasing, then:* $\lim_{n\to\infty} a_n =$ ______________.
- *If it is decreasing, then:* $\lim_{n\to\infty} a_n =$ _____________.

Example 5.45 Given the following increasing bounded (above) sequence, find its limit:

$$a_1 = 0,\ a_{n+1} = a_n^2 + \frac{1}{4}.$$

The following proposition and theorem are very strong when it comes to figuring out the limits of sequences.

Proposition 5.46 *Given two convergent sequences $\{s_n\}$ and $\{t_n\}$ and a number $c \in \mathbb{R}$, then*

(1) $\lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$.

(2) $\lim_{n\to\infty} cs_n = c\lim_{n\to\infty} s_n$.

(3) $\lim_{n\to\infty} s_n \cdot t_n = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} t_n$.

Theorem 5.47 (Squeeze theorem) *If we have three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ such that*

$$a_n \leq b_n \leq c_n$$

and if there exists L such that

$$\lim_{n\to\infty} a_n = L \text{ and } \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = L.$$

Example 5.48 Use the squeeze theorem to show that the sequence $\{\frac{1}{n+5}\}$ converges to 0.

Example 5.49 Let's relook at Problem 5.38 with sequence:

$$\{c_n\} = \left\{\frac{5n+3}{n+4} + \frac{3n^2-7}{n^2+8}\right\}$$

Exercise 5.50 At home, show the two things that we skipped over above:

$$\lim_{n\to\infty} \frac{-17}{n+4} = 0 \text{ and } \lim_{n\to\infty} \frac{-31}{n^2+8} = 0$$

Problem 5.51 With a partner, calculate the limit of the following sequence using the squeeze theorem.

$$\lim_{n\to\infty} \frac{\sin(n)}{n}$$

Order of growth rate

The following is useful for using the squeeze theorem.

- n^n grows faster than $n!$.
- $n!$ grows faster than a^n with $a > 1$.
- a^n grows faster than n^k where $k \in \mathbb{N}$.
- A polynomial grows at the rate of its degree ($n < n^2 < n^3 < \dots$)

Together we have:

$$\log(n) < n < a^n < n! < n^n$$

6 Series

6.1 Definitions

Remember that a sequence $a = \{a_n\}$ is a map which assigns to each $n \in \mathbb{N}$ a real number and that a_n is a shorthand for $a(n)$.

Definition 6.1 An *infinite series* is the sum of ____ terms of a sequence.

Wikipedia: infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$$

Example 6.2

$$\sum_{n=1}^{\infty} \frac{1}{n} =$$

Definition 6.3 Given an infinite series, the sum of the first N terms is called the *partial sum* and is denoted by S_N.

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_N$$

Example 6.4 Given the series

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \ldots$$

The first four partial sums are:

$$\begin{aligned} S_1 &= \underline{\quad} \\ S_2 &= \underline{\hspace{6cm}} \\ S_3 &= \underline{\hspace{8.5cm}} \\ S_4 &= \underline{\hspace{12cm}} \end{aligned}$$

6.2 Convergence

Wikipedia: convergent

Definition 6.5 If the sequence of partial sums converges to a number S, we say that the series is *convergent* and its sum is S. In other words, if there exists S such that $\lim_{N\to\infty} S_N = S$ then $\sum_{n=1}^{\infty} a_n = S$.

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \sum_{n=1}^{\infty} a_n = S$$

Example 6.6 Let's find the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

The partial sums are:

$$\begin{aligned} S_1 &= \underline{\quad} \\ S_2 &= \underline{\hspace{4cm}} \\ S_3 &= \underline{\hspace{7cm}} \\ S_N &= \underline{\hspace{14cm}} \end{aligned}$$

The N-th element of the sequence of partial sums can be written as

$$1 - \frac{1}{2^N}.$$

The limit of this sequence is:

Therefore, the sum of the infinite series is 1.

Definition 6.7 If the sequence of the partial sums is divergent or oscillating, then we say that the series is *divergent* or *oscillating*.

Example 6.8 Consider the infinite series:

$$\sum_{n=1}^{\infty}(-1)^n$$

with sequence of partial sums:

$$\{S_N\} = -1,\, 0,\, -1,\, 0,\, -1,\, \ldots$$

Then what does the infinite sum equal?

$$\lim_{N\to\infty} S_N = ?$$

____

____

____________________________________ In other words,

$$\sum_{n=1}^{\infty}(-1)^n \;\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Proposition 6.9 (Necessary condition for convergence) *If* $\sum_{n=1}^{\infty} a_n$ *converges then* $\lim_{n\to\infty} a_n = \_\_$.

If the limit $\lim_{n\to\infty} a_n$ *does not exist, or has a value different than* $\_\_$*, then the series diverges.*

Proof. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then,

$$\lim_{N\to\infty} S_N = L \text{ and } \lim_{N\to\infty} S_{N-1} = L$$

Furthermore,

$$\lim_{N\to\infty}(S_N - S_{N-1}) = L - L = 0.$$

The partial sums are:

$$S_N = a_1 + a_2 + \ldots + a_N$$
$$S_{N-1} = a_1 + a_2 + \ldots + a_{N-1}$$

Then,

$$S_N - S_{N-1} = (a_1 + a_2 + \ldots + a_N) - (a_1 + a_2 + \ldots + a_{N-1}) = a_N$$

In other words

$$\lim_{N\to\infty}(S_N - S_{N-1}) = \lim_{N\to\infty} a_N = 0$$

as desired. □

If $\lim_{n\to\infty} a_n = 0$ we can **not** state that the series converges.

Example 6.10 Consider the following infinite series

$$\sum_{n=1}^{\infty} \frac{n}{n+1}.$$

Then

$$\lim_{N\to\infty} \frac{N}{N+1} = \underline{\hspace{3cm}}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

is ________________.

Definition 6.11 If $\lim_{n\to\infty} a_n = 0$, then the sequence $\{a_n\}$ is called a *null sequence*.

6.3 Properties

Proposition 6.12 *Given $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_i$, two infinite series with sums respectively A and B, and given $\lambda \in \mathbb{R}$.*

Then,

(1) $\sum_{i=1}^{\infty} \lambda a_i = \underline{\hspace{1.5cm}}$.

(2) $\sum_{i=1}^{\infty} (a_i + b_i) = \underline{\hspace{3.5cm}}$.

Proof. We show the sum obtained after multiplication by a constant $\lambda \in \mathbb{R}$.

The partial sums of the infinite series $\sum_{n=1}^{\infty} \lambda a_n$ are:

$$\begin{aligned}
S_1' &= \lambda a_1 \\
S_2' &= \lambda a_1 + \lambda a_2 \\
S_3' &= \lambda a_1 + \lambda a_2 + \lambda a_3 \\
S_N' &= \lambda a_1 + \lambda a_2 + \lambda a_3 + \ldots + \lambda a_N
\end{aligned}$$

Then,

$$S_N' = \lambda(a_1 + a_2 + a_3 + \ldots + a_N) = \lambda S_N$$

and therefore

$$\lim_{N\to\infty} S_N' = \lim_{N\to\infty} \lambda S_N.$$

By definition,

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = A$$

If $\lambda \in \mathbb{R}$, then $\lim_{N\to\infty} \lambda S_N = \lambda \lim_{N\to\infty} S_N$.

Thus, $\lim_{N\to\infty} S'_N = \lambda A$ and the sum $\sum_{n=1}^{\infty} \lambda a_n = \lambda A$. $\square$

6.4 Geometric series

Definition 6.13 A *geometric series* is a series of the form:

Wikipedia: geometric series

$$\sum_{n=1}^{\infty} \qquad (2.1)$$

where $a, r \in \mathbb{R}$.

Proposition 6.14 *Given a geometric series, the N-th partial sum is given by the equation:*

$$S_N = \qquad .$$

Therefore, the sum of the series is

$$\sum_{n=1}^{\infty} \qquad = \lim_{N\to\infty} \qquad (2.2)$$

such that it converges when ____________ *and diverges when* ____________.

If the series converges, then its sum is:

$$\sum_{n=1}^{\infty} ar^{n-1} = \qquad (2.3)$$

Proof. The N-th partial sum of a geometric series is:

$$S_N = a + ar + ar^2 + ar^3 + ... + ar^{N-1}$$

Multiplying both sides by r gives:

$$rS_N = ar + ar^2 + ar^3 + ... + ar^N$$

Subtracting the second equation from the first, we have:

$$\begin{aligned} S_N - rS_N &= a - ar^N \\ (1-r)S_N &= a - ar^N \\ S_N &= \frac{a - ar^N}{1-r} \end{aligned}$$

A series converges if $\lim_{N\to\infty} S_N$ exists. Then if $|r| < 1$, the limit is convergent. □

Example 6.15 Given the series defined by the partial sums:

$$S_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^N}$$

This series is geometric, with $a =$ ___ and $r =$ ___.

We have that $r < 1$, therefore the series is convergent with sum:

$$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1.$$

6.5 Harmonic series

Wikipedia: harmonic series

Definition 6.16 The *harmonic series* is the series

$$\sum_{n=1}^{\infty} \quad .$$

Remember that the limit of the sequence being 0 does not imply that the series converges? The harmonic series is our counterexample! It diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$. Let's see why it diverges:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ldots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \\ &= \infty\end{aligned}$$

6.6 p-series

In order to make the harmonic series converge, we can increase the power of the denominator.

Definition 6.17 A *p-series* is a series of the form

$$\sum_{n=1}^{\infty}$$

where $p \in \mathbb{R}$. It converges when ________ and diverges when ________.

6.7 Telescopic series

Definition 6.18 A series is *telescoping* if it is of the form:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}$$

Proposition 6.19 *The N-th partial sum of the series is given by:*

$$S_N = \qquad .$$

Therefore, the sum of the series is:

$$S = \lim_{N\to\infty} b_{N+1} - b_1.$$

Proof. Given an infinite series whose terms a_n can be written in the form $b_{n+1} - b_n$.

The N-th partial sum is given by:

$$S_N = a_1 + a_2 + \cdots + a_N = b_2 - b_1 + b_3 - b_2 + \ldots + b_{N+1} - b_N = b_{N+1} - b_1.$$

The limit of the n-th term of the sequence of partial sums is:

$$S = \lim_{N\to\infty} b_{N+1} - b_1$$

□

Example 6.20 Let's find the sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

We can separate the quotient in two:

Setting n to be $\frac{1}{2}$ implies $A = \frac{1}{2}$ and setting $n = -\frac{1}{2}$ gives $b = -\frac{1}{2}$.

In other words:

$$\frac{1}{(2n-1)(2n+1)} = \frac{1/2}{2n-1} - \frac{1/2}{2n+1}.$$

We see that this is telescoping since plugging in $n+1$ into the first term gives:

$$\frac{1/2}{2(n+1)-1} = \frac{1/2}{2n+1}$$

implying:

$$b_n = \frac{1/2}{2n-1}$$

giving us our telescoping sum.

Then,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \sum_{n=1}^{\infty}$$

The limit of the N-th partial sum is

$$\begin{aligned} S &= - \lim_{N\to\infty} b_{N+1} - b_1 \\ &= - \lim_{N\to\infty} \frac{1/2}{2N+1} - \frac{1/2}{2(1)-1} \\ &= - \lim_{N\to\infty} \frac{1/2}{2N+1} - \frac{1}{2} \\ &= \lim_{N\to\infty} \underline{\quad} \end{aligned}$$

Therefore, the sum of the infinite series is ___.

6.8 Arithmetico-geometric series

Wikipedia: arithmetico-geometric

Definition 6.21 A series is *arithmetico-geometric* if it has the form:

$$\sum_{n=0}^{\infty}$$

where $c, d, r \in \mathbb{R}$.

Proposition 6.22 *The N-th partial sum of an arithmetico-geometric series is:*

$$S_N = d + (c+d)r + (2c+d)r^2 + \cdots + (cN+d)r^N$$

The sum of the series is given by:

$$\sum_{n=0}^{\infty}(cn+d)r^n =$$

and converges when ___________ and diverges when ___________.

In the case where $c = 0$, then the series is geometric.

Proof. An arithmetico-geometric series has the form

$$\sum_{n=0}^{\infty}(cn+d)r^n$$

The N-th partial sum of the series is:

$$S_N = d + (c+d)r + (2c+d)r^2 + \cdots + (cN+d)r^N$$

Combining like terms, multiplying by r and subtracting one from the other:

$$\begin{aligned} S_N &= d + dr + dr^2 + \cdots + dr^N + cr + 2cr^2 + \cdots + Ncr^N \\ rS_N &= dr + dr^2 + dr^3 + \cdots + dr^{N+1} + cr^2 + 2cr^3 + \cdots + Ncr^{N+1} \\ S_N - rS_N &= d - dr^{N+1} + cr + cr^2 + \cdots + cr^N - Ncr^{N+1} \end{aligned}$$

Defining $t_N = cr + cr^2 + \cdots + cr^N$ then gives

$$\begin{aligned} t_N - rt_N &= cr - cr^{N+1} \\ t_N &= \frac{cr - cr^{N+1}}{1-r} \end{aligned}$$

Therefore,

$$\begin{aligned} S_N - rS_N &= d - dr^{N+1} + \frac{cr - cr^{N+1}}{1-r} - Ncr^{N+1} \\ S_N &= \frac{d - dr^{N+1}}{1-r} + \frac{cr - cr^{N+1}}{(1-r)^2} - \frac{Ncr^{N+1}}{1-r} \end{aligned}$$

Taking the limit as N goes to infinity gives

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{d - dr^{N+1}}{1-r} + \frac{cr - cr^{N+1}}{(1-r)^2} - \frac{Ncr^{N+1}}{1-r} \right)$$

This limit only converges when $|r| < 1$ with solution

$$S = \frac{d}{1-r} + \frac{cr}{(1-r)^2}.$$

Therefore,

$$\sum_{n=0}^{\infty} (cn+d)r^n = \frac{d}{1-r} + \frac{cr}{(1-r)^2}$$

when $|r| < 1$. □

Example 6.23 Let's find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{n}{3^n} =$$

This is an arithmetico-geometric series with $c =$ __, $d =$ __, and

$r = \underline{\quad}$. And since $|r| < 1$, it is convergent and converges to

$$\sum_{n=0}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} =$$

7 Convergence

There are many ways to prove convergence or to test for convergence.

7.1 Change of index

Sometimes it is necessary to change the first index of a series (for example, to use a known definition).

Here we consider two methods:

Method 1 We can include a new element (or lost element), and kept it in mind when we calculate the sum.

Example 7.1 Find the sum of the infinite series:

$$\sum_{n=1}^{\infty} (2n+5)\frac{1}{3^n} = \frac{7}{3} + 1 + \frac{11}{27} + \dots$$

an arithmetico-geometric series, whose definition requires that the index start at $n = 0$.

We change the index and see its terms:

$$\sum_{n=0}^{\infty} (2n+5)\frac{1}{3^n} = 5 + \frac{7}{3} + 1 + \frac{11}{27} + \dots$$

Therefore,

$$\sum_{n=1}^{\infty} (2n+5)\frac{1}{3^n} = -5 + \sum_{n=0}^{\infty} (2n+5)\frac{1}{3^n}$$

We resolve the new series with the known definition and subtract the new element to obtain the original sum.

Method 2 Substituting n with $n+1$ in the definition of the series.

Example 7.2

$$\sum_{n=1}^{\infty} (2n+5)\frac{1}{3^n} = \sum_{n=0}^{\infty} (2(n+1)+5)\frac{1}{3^{n+1}}$$

To change $n = 0$ to the original $n = 1$, we must add 1. Therefore, we replace n with $n+1$.

Now we simplify the new series

$$\begin{aligned}\sum_{n=1}^{\infty}(2n+5)\frac{1}{3^n} &= \sum_{n=0}^{\infty}(2n+7)\frac{1}{3}\frac{1}{3^n} \\ &= \sum_{n=0}^{\infty}\frac{2n+7}{3}\frac{1}{3^n} \\ &= \frac{7}{3}+1+\frac{11}{27}+\dots\end{aligned}$$

7.2 Absolute convergence

Definition 7.3 A series $\sum a_n$ is *absolutely convergence* if __________ is convergence. A convergent series $\sum a_n$ such that $\sum|a_n|$ diverges is called *conditionally convergent*.

Theorem 7.4 (Absolute convergence test) *If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.*

7.3 Comparison test

Proposition 7.5 (Comparison test) *Given two sequences $\{a_n\}$ and $\{b_n\}$. If there exists $m \in \mathbb{N}$ such that for all $n > m$ we have $0 \leq a_n \leq b_n$, then*

Wikipedia: *Comparison test*

$$\begin{aligned}\sum_{n=1}^{\infty} b_n \text{ convergent} \quad &\Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ \sum_{n=1}^{\infty} a_n \text{ divergent} \quad &\Rightarrow \quad \sum_{n=1}^{\infty} b_n \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_}\end{aligned}$$

Example 7.6 Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{1}{2^n+1}$$

We know

$$\frac{1}{2^n+1} < \qquad \qquad \forall n \in \mathbb{N}.$$

7.4 Limit comparison test

Proposition 7.7 (Limit comparison test) *Given two series $\sum a_n$ and $\sum b_n$ with positive terms and let ________________. If $L > 0$ and finite then both series converge or both series diverge. If $L = 0$ and*

Wikipedia: *Limit comparison test*

$\sum b_n$ converges, then $\sum a_n$ converges. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example 7.8 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 3\sqrt[4]{n} + 1}$$

We compare with the following series.

Then we have:

The series $\sum b_n$ is a p-series with $p = \frac{1}{2}$. Since $p \leq 1$, this series is divergent. Therefore, by the limit comparison test, $\sum a_n$ is also divergent.

7.5 Ratio test

Wikipedia: Ratio test

Proposition 7.9 (Ratio test) *Given an infinite series $\sum_{n=1}^{\infty} a_n$, let*

Then,

- *If $\lambda < 1$ then $\sum a_n$ ________________.*
- *If $\lambda > 1$ then $\sum a_n$ ______________.*
- *If $\lambda = 1$ or the limit does not exist, then the test is inconclusive.*

Example 7.10 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n+1}{n!}$$

Then,

Therefore by the ratio test, since $\lambda < 1$, the series converges.

7.6 Root test

Proposition 7.11 (Root test) *If $\sum a_n$ is an infinite series with positive terms, and*

Wikipedia: *Root test*

Then,

- *If $\lambda < 1$ then $\sum a_n$ ________________.*
- *If $\lambda > 1$ then $\sum a_n$ ______________.*
- *If $\lambda = 1$ or the limit does not exist, then the test is inconclusive.*

Example 7.12 Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{(\log(n))^n}$$

We try and apply the root test:

Therefore, by the root test, since $\lambda < 1$, then $\sum a_n$ converges.

7.7 Alternating series test

Definition 7.13 A series is *alternating* if it is of the form:

Wikipedia: alternating

$$\sum_{n=0}^{\infty} \qquad .$$

Proposition 7.14 (Alternating series test) *Given an alternating series $\sum(-1)^n a_n$, if the sequence $\{|a_n|\}$ is decreasing such that*

Wikipedia: *Alternating series test*

$$\lim_{n\to\infty} a_n = 0$$

then, the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ *converges.*

Example 7.15 Determine if the following alternating series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^3+1}$$

Trying to apply the ______________________________ test, we look at the limit of the terms.

It remains to see if $|a_n|$ is decreasing or not, *i.e.*, if $|a_{n+1}| < |a_n|$ for all n.

Therefore $|a_n|$ is decreasing and by the alternating series test $\sum(-1)^n a_n$ is convergent.

7.8 More examples

Example 7.16 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \right)^2 .$$

Example 7.17 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}.$$

Example 7.18 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(1-\sin(n))(1+\sin(n))}{n^2+8n+1}.$$

Example 7.19 Determine if the following series converges or diverges. If it converges, determine if it converges absolutely or conditionally.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+1}{n(n+1)}.$$

Problem 7.20 With a partner, determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{10^{3+5n}}{n^{1+n^2}}$$

Exercise 7.21 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$$

Chapter 3

Limits and continuity

8 Bounded functions

Definition 8.1 If $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ is a one-variable real function, we say that f is a *bounded function* if there exists $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in A$.

Example 8.2 The function $f(x) = \sin(x)$ is bounded between _____ and __.

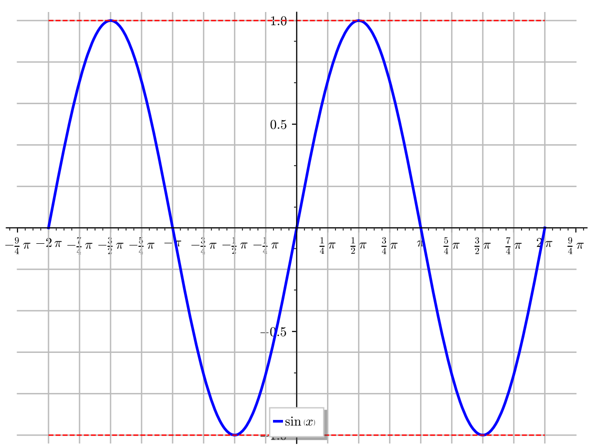


Figure 3.1: Bounds of $f(x) = \sin(x)$

Definition 8.3 The *supremum* of a function $f(x)$ is the minimum of the set of its upper bounds.

$$\sup_{x \in A} f(x) = \sup \{ f(x) \mid x \in A \}$$

Definition 8.4 The *infimum* of a function $f(x)$ is the maximum in the set of all lower bounds.

$$\inf_{x \in A} f(x) = \inf \{ f(x) \mid x \in A \}$$

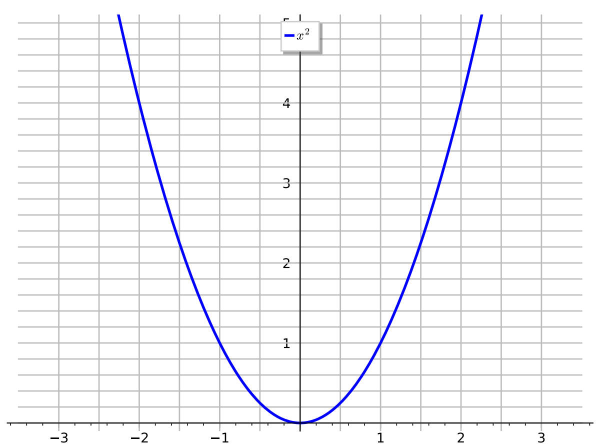
Example 8.5 Find the bounds of the function $f(x) = \sin(4x) - 1$.

The infimum of $f(x)$ is _____ and the supremum is __.

9 Limits

9.1 A tangential problem

Problem 9.1 Suppose we have the function $f(x) = x^2$. How do we find the slope of the tangent line at $x = 1$?



With a partner, find the slope of the tangent line at

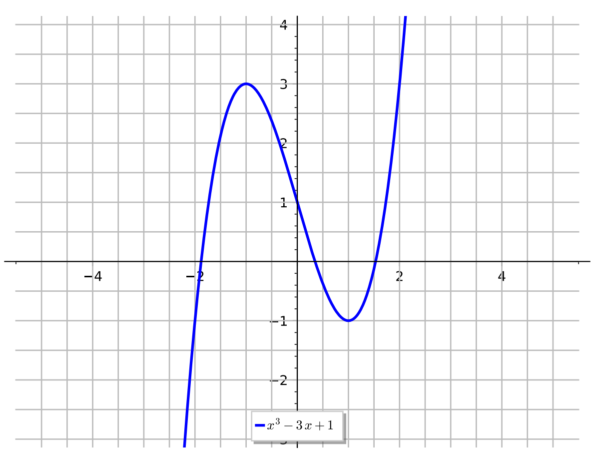
- $x = 0$
- $x = 2$
- $x = -1$

Can you and your partner come up with a formula for the slope of the tangent line for arbitrary x?

9.2 Limits

What we just did can be formalized into what's called limits. We want to find the value of a function at a certain point without knowing the value at that point exactly.

Example 9.2 Let's first start with an example to help clarify. Let $f(x) =$ ____________________.

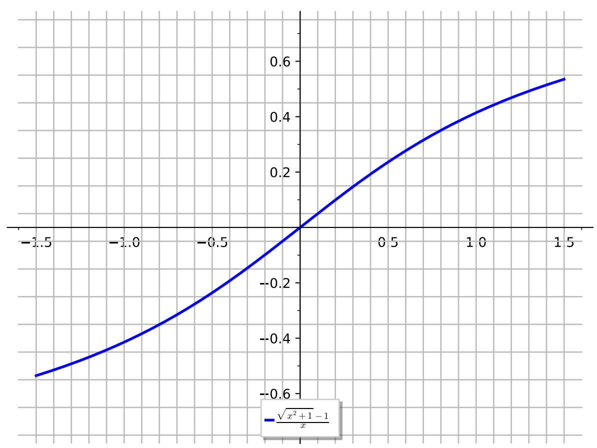


Try and calculate $f(\frac{3}{2})$ without plugging in the number.

x	f(x)	x	f(x)
1		2	
1.4		1.6	
1.49		1.51	
1.499		1.501	
1.4999		1.5001	
1.49999		1.50001	

It looks like the value is approaching ________________________. We write this as:

Problem 9.3 With a partner find the limit of the function $f(x) = \frac{\sqrt{x^2+1}-1}{x}$ as x approaches ___. (Use a calculator!) Here's the graph:



x	f(x)	x	f(x)
-1		1	
-0.5		0.5	
-0.1		0.1	
-0.01		0.01	
-0.001		0.001	
-0.0001		0.0001	

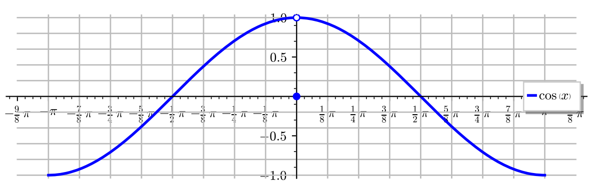
What do you both think is the limit?

Exercise 9.4 At home, try and find the limit of $f(x) = \frac{2x-2}{x^3-1}$ at $x = 1$.

Example 9.5 Suppose we have the following piecewise function

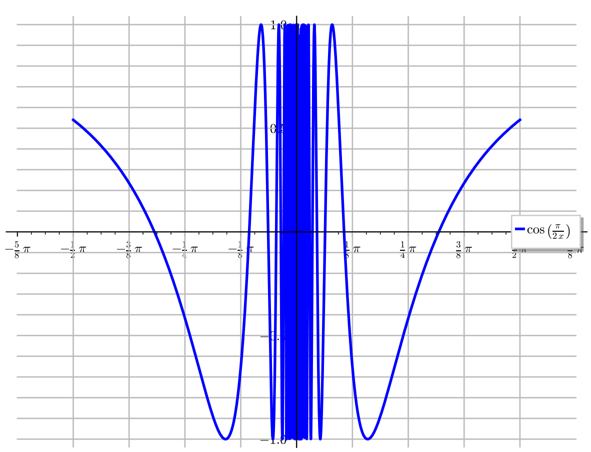
$$f(x) = \begin{cases} \cos(x) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and we want to know the limit at $x = 0$. Using the graph of the function, what do you think the limit is?



$$\lim_{x\to 0} f(x) = \_\_$$

Example 9.6 Let's look at something a little more complex. Say we have the function $f(x) = \cos(\frac{\pi}{2x})$ whose graph is the following:



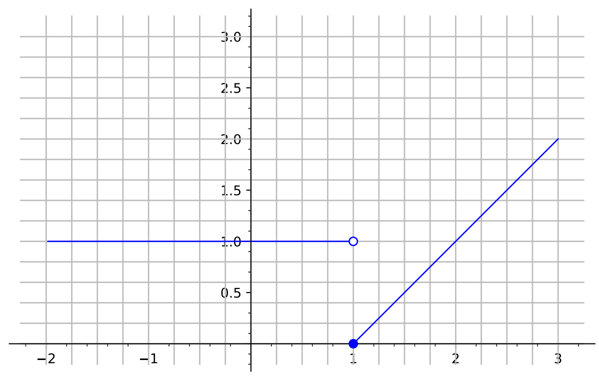
What is the limit at $x = 0$?

$$\lim_{x\to 0} f(x) = \underline{\hspace{5cm}}$$

What happens if our function is split in two? Let

$$f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 1 \end{cases}$$

whose graph is given by:

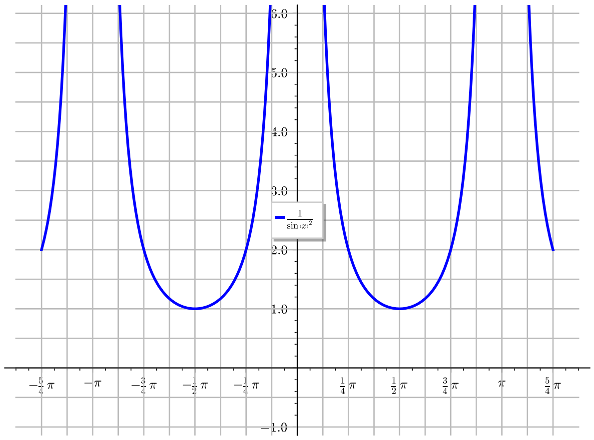


How can we calculate the limit at $x = 1$? What we can do is look at the limit from only one side instead of both sides like we had been doing. There are two sides of $x = 1$, a positive side and a negative side. Looking at the limit from the positive side, it looks like the limit is equal to __. Looking at the limit from the negative side, it looks like the limit is equal to __. We denote these by

Comparing these limits with our original definition of limit, we see

$\lim_{x\to a} f(x) = L$ if and only if ____.

Let's look at another example. Suppose $f(x) =$ ____________ with the following graph:

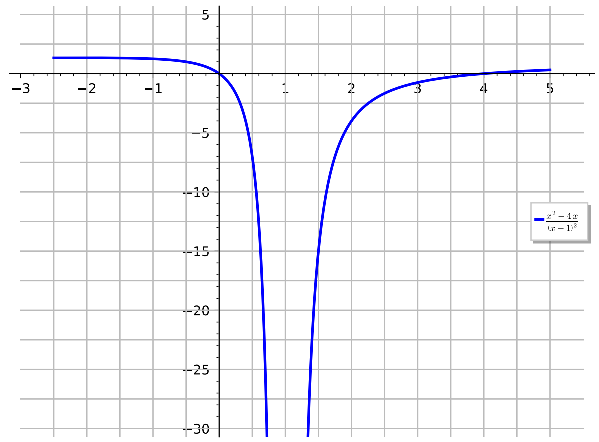


What is the limit as x approaches 0? The closer x gets to 0 (from either side) $\sin(x)$ gets closer to 0 and therefore $\frac{1}{\sin^2(x)}$ gets larger and larger. In fact, the limit does not approach any particular number, but keeps growing! To represent this, we say that the limit goes to infinity:

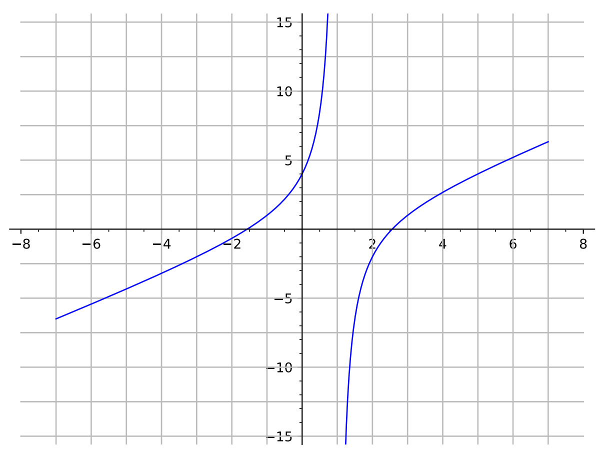
This does NOT mean that ∞ is a number! Nor are we saying that the limit exists. We are merely saying that as we approach $x = 0$, our function gets larger and larger and larger, without end.

Similarly, if the function $f(x)$ gets smaller and smaller as we approach a point a, then we can say that the limit is negative infinity:

Example 9.7 Let $f(x) = \frac{x^2-4x}{(1-x)^2}$ then as $x \to 1$ the limit goes to ____________________________.



Exercise 9.8 With a partner, let $f(x) = \frac{x^2-x-4}{x-1}$ whose graph is given by:



What are the following limits:

$$\lim_{x\to 1^-} f(x) = \_\_\_\_ \qquad \lim_{x\to 1^+} f(x) = \_\_\_\_\_\_\_\_ \qquad \lim_{x\to 1} = \_\_\_\_\_\_\_\_\_\_$$

9.3 Precise definition of limit

Let's be a little more precise by what we mean by a limit.

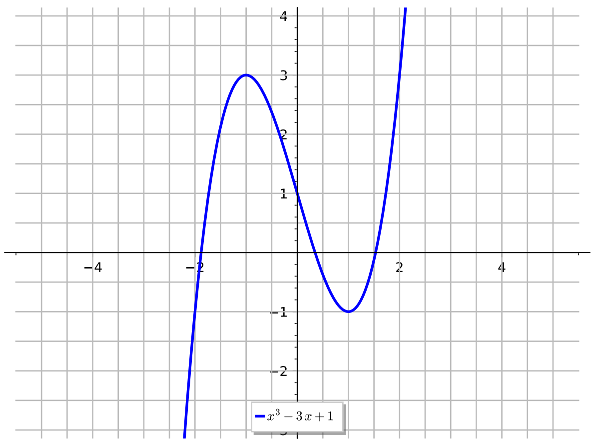
Definition 9.9 (Precise definition of a limit) Let f be a function defined on some open interval that contains a, but not necessarily defined at a. Then the *limit of $f(x)$ as x approaches a is L*, written

____

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \text{ then } |f(x) - L| < \varepsilon$$

What does this mean?



We can use this definition as a way to show whether or not the intuitive solution to a limit problem is accurate or not. In order to prove that our intuition is correct, we use what's called an *epsilon-delta proof*. The following is a general outline of how an epsilon-delta proof works:

Epsilon-delta proof

Analysis (1) ____
____

(2) ____
____

Proof (1) ____

(2) ____
____

(3) ____

Finish Finish up the proof.

Example 9.10 (Epsilon-delta proof) We want to prove that $\lim_{x \to 2}(3x + 2) = 8$.

Analysis (1) We start with $|f(x) - L| < \varepsilon$. In our case, $L =$ __ and $f(x) =$ ___________. Therefore

(2) We want to end with something like _________________________.

Proof (1) Suppose _________.

(2) Let _________________.

(3) If _________________________, then

Finish Therefore, by the definition of a limit

Problem 9.11 With a partner, prove that $\lim_{x\to 1}(5x-2) = 3$.

Example 9.12 Let's do a significantly harder example. Prove that $\lim_{x\to 2} x^2 + x - 2 = 4$.

Analysis (1)

(2)

Proof (1)

(2)

(3)

Finish

Definition 9.13 (Left-hand and Right-hand limits) We say

We use the exact same format to prove things in this case.

Example 9.14 Prove that $\lim_{x\to 8^+} \sqrt[3]{x} = 2$.

Analysis (1)

(2)

Proof (1) ____________________________

(2) ____

(3)

Finish

One last precise limit definition!

Definition 9.15 (Infinite limits) Let f be a function defined on some open interval that contains the number a, except potentially not defined at a itself.

Example 9.16 Prove $\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$.

Analysis (1)

(2)

Proof (1) ____________________________.

(2) ______________________________.

(3)

Finish Therefore, by definition of the infinite limit:

9.4 Limit laws

In a lot of cases, the best way to know the limit is to plug in the number directly.

Theorem 9.17 *If f is a polynomial or a rational function and a is in the domain of f, then*

$$\lim_{x\to a} f(x) = \underline{\hspace{3em}}$$

If we know the limit to a few functions at a point, then we can use that information to figure out the limit of their combinations!

Theorem 9.18 *Suppose c is some constant and suppose the following limits exist:*

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

Then

(1)

$$\lim_{x\to a} (f(x) + g(x)) = \underline{\hspace{14em}}$$

(2)

$$\lim_{x\to a} (cf(x)) = \underline{\hspace{7em}}$$

(3)

$$\lim_{x\to a} (f(x)g(x)) = \underline{\hspace{16em}}$$

Another important rule helps us when one function is extremely similar to another function.

Theorem 9.19 *If $f(x) = g(x)$ when $x \neq a$ and if $\lim_{x\to a} f(x)$ exists, then*

We saw an example of this in Example 9.5.

Two more useful properties of limits are given next.

Theorem 9.20 *If $f(x) \leq g(x)$ when x is near a (except possibly at a), and the limits of f and g both exist as x approaches a, then*

Theorem 9.21 (The squeeze theorem) *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and*

Wikipedia: Squeeze theorem

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{xtoa} g(x) = L$$

The squeeze theorem is a super difficult theorem to use, but it is sometimes the easiest method of finding a limit!

Example 9.22 Let

$$f(x) = x^3 \cos\left(\frac{\pi}{2x}\right).$$

Show that $\lim_{x \to 0} f(x) = 0$. It would be super nice if we could just use one of our limit rules, but we found out in Example 9.6 that $\lim_{x \to 0} \cos\left(\frac{\pi}{2x}\right)$ does not exist. So, instead we're going to have to use the squeeze theorem.

One thing to notice is that cos always oscillates between -1 and 1. In other words:

$$-1 \leq \cos\left(\frac{\pi}{2x}\right) \leq 1$$

Multiply all sides by x^3 and we have

Now we have something that looks like the squeeze theorem! Since both x^3 and $-x^3$ are polynomials and 0 is in their domains we have:

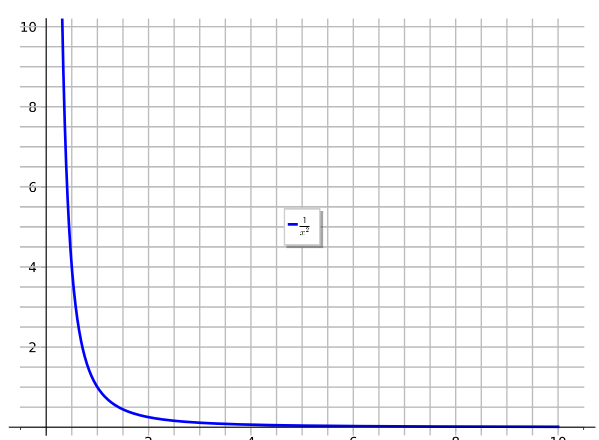
And, by the squeeze theorem (!) we know that

Exercise 9.23 With a partner, use the squeeze theorem to show that

9.5 To infinity and beyond

Although infinity is not a number, we can look at limits as a function gets really really big, *i.e.*, when the function goes toward infinity.

Example 9.24 Let $f(x) = \frac{1}{x^2}$ and notice that this function is always positive. Here is a graph of the function:



What this means is that the higher the value of x, the closer to __ we are going to get. Intuitively we say that the limit of $f(x)$ as x goes to infinity is 0. We write this as:

We can play this game with any function. If f is a function defined on (a, ∞) then

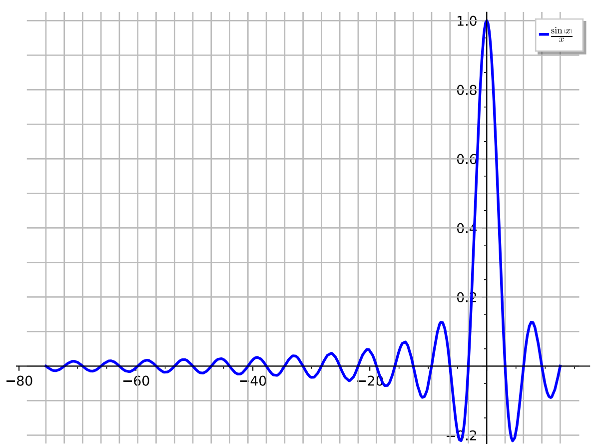
$$\lim_{x \to \infty} f(x) = L$$

means that the value of $f(x)$ approaches L as x gets larger and larger.

Similarly, you can do the same thing for negative infinity.

$$\lim_{x \to -\infty} f(x) = L$$

Example 9.25 Let $f(x) = \frac{\sin(x)}{x}$ with the graph:



What is the solution to the following?

$$\lim_{x \to -\infty} f(x) = \_\_$$

The line $y = L$ is called a *horizontal asymptote* of the curve $f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

There is a battle in mathematics on the definition of a horizontal asymptote! The majority give the definition above, but some authors additionally require that the line $f(x)$ doesn't cross the asymptote!

Problem 9.26 Let $f(x) = \frac{1}{x}$. With a partner, find

$$\lim_{x \to \infty} f(x) = \_\_ \text{ and } \lim_{x \to -\infty} f(x) = \_\_$$

What do you think the following theorem should say?

Theorem 9.27 *If $r > 0$ is a rational number such that x^r is defined for all x, then*

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = \underline{\quad}$$

This theorem is ridiculously strong! Let's look at an example.

Example 9.28 Let $f(x) = \frac{4x^5+3x^2+x-77}{7x^5-6x^3+22x^2-943}$. What is

$$\lim_{x\to\infty} f(x) = \underline{\quad}?$$

Example 9.29 Evaluate

$$\lim_{x\to\infty} 2x - \sqrt{4x^2+1}.$$

Example 9.30 Evaluate

$$\lim_{x\to\infty} \cos(x)$$

Example 9.31 Evaluate

$$\lim_{x\to\infty} x$$

Example 9.32 Evaluate

$$\lim_{x\to\infty} x^3 - x^2$$

Problem 9.33 With a partner try and evaluate

$$\lim_{x\to\infty} \frac{x^2 - x}{2x - 5}$$

Definition 9.34 (Precise definition of limit at infinity) Let f be a function defined on the interval (a, ∞) for some number a. Then,

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

Let f be a function defined on the interval $(-\infty, a)$ for some number a. Then,

$$\lim_{x\to-\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

Let f be a function defined on the interval (a, ∞) for some number a. Then,

$$\lim_{x\to\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

10 Continuity

Limits are super easy when we can just plug in a number to get the answer. When we can just plug in a number a into a function $f(x)$ in order to get the limit we say that the function is continuous at a.

Definition 10.1 A function f is *continuous at a* if

For this equation to hold there are three things that normally need to be proved:

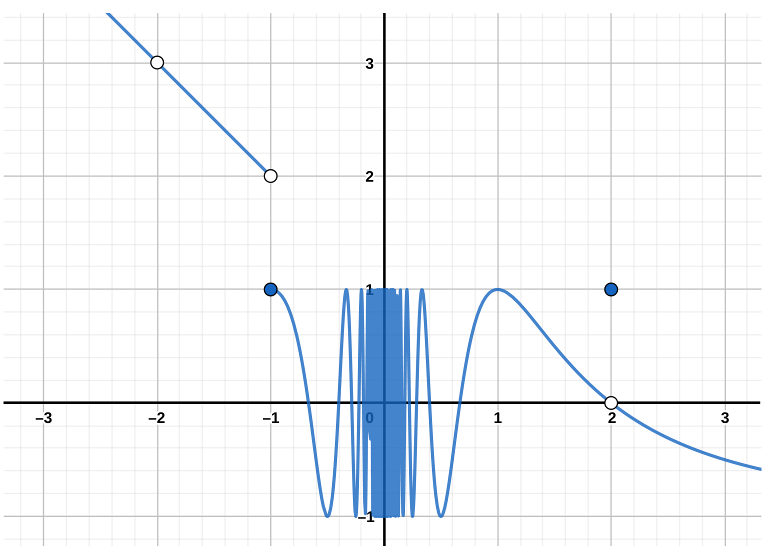
(1) ____________________________

(2) ___________________________________

(3) _____________________________________

If f is not continuous at a we say that it is *discontinuous at a*.

Example 10.2 This example shows us the four different types of discontinuity that exists.



(1) ____

(2) ____

(3) ____

(4) ____

We can similarly define left-hand and right-hand continuity.

Definition 10.3 A function f is *continuous from the left at a* if

A function f is *continuous from the right at a* if

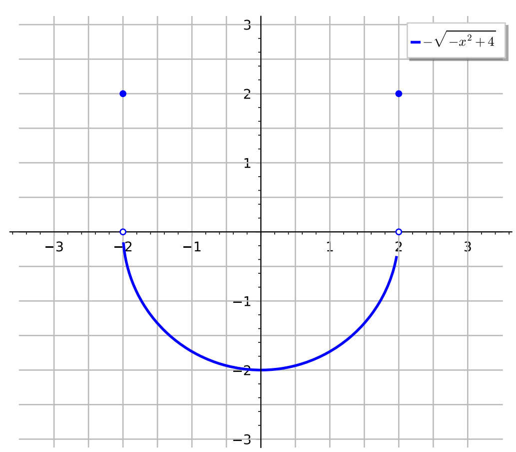
Similarly, we can do it on an interval!

Definition 10.4 A function f is *continuous on an interval I* if it is continuous at every number in the interval.

Example 10.5 Suppose

$$f(x) = \begin{cases} -\sqrt{4-x^2} & \text{if } -2 < x < 2 \\ 2 & \text{if } x = -2 \text{ or } x = 2 \end{cases}$$

Show that f is continuous on the interval $(-2, 2)$, but discontinuous everywhere else. Here is a graph of the function:



Just like the limit laws, we have rules for continuity.

Theorem 10.6 *If f and g are continuous functions at a and c is some constant, then the following functions are also continuous at a:*

(1) __________

(2) ____

(3) ____

There are a lot of functions which are continuous!

(1) All ____________________ functions are continuous everywhere.

(2) _______________ functions are continuous on their domain.

(3) _______ functions are continuous on their domain.

(4) ____________________ functions are continous on their domain.

(5) _____________________ functions are continuous on their domain.

(6) _____________________ functions are continuous on their domain.

Example 10.7 Where is the function $f(x) = \frac{x^3+2x^2-1}{71x-3}$ continuous?

Exercise 10.8 With a partner, state where $f(x) = \frac{\ln(x)}{x^2-4}\frac{\ln(x)}{x^2-4}$ is continuous?

What about composition $f \circ g$?

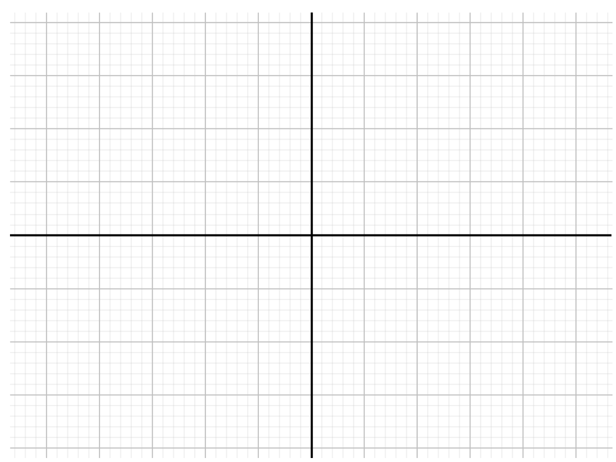
Theorem 10.9 *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then*

If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

Continuous functions have a nice property in that they go through every number.

Theorem 10.10 (Intermediate value theorem) *If f is continuous on the interval $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = N$.*

This is best visualized through a picture.



This theorem is super helpful for finding roots to polynomials!

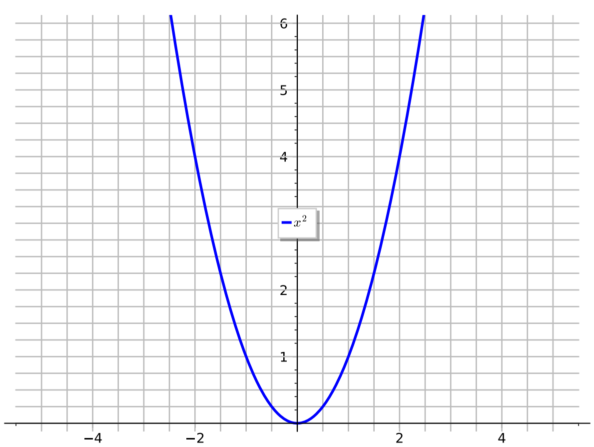
Chapter 4

Derivatives of one variable

11 Derivatives

Remember how at the beginning of limits we looked at a tangent line to a curve. We can now use all the technology we've developed to answer this question for any arbitrary curve!

Let's look at that very first exercise. Let $f(x) = x^2$, what is the slope of the tangent line at $x = 1$?



Recall that the method we used was to find secants to the curve. So we used equations like:

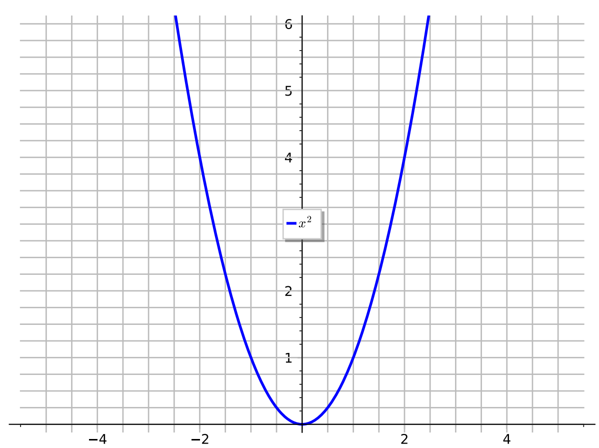
If we keep moving x closer and closer to a then we will eventually find the tangent line. In other words, the tangent line at point $(a, f(a))$

has as slope:

and from there you can figure out the tangent line for any given point!

Problem 11.1 Use this method to find the tangent line $y = mx + b$ when $a = 3$ with a partner.

There is a second way to find the slope. We can fix a distance away from the point a, and they try and decrease this distance. What does this mean?



Start with some point a where you want to find the tangent line. You let h be some distance away from the point a. Then you try and decrease the distance and solve the following limit:

Sometimes, this method is easier to solve than the previous version.

Example 11.2 Let $f(x) = 1/x^2$ and find the slope of the tangent line at $(2, \frac{1}{4})$.

This slope line has a special name, and it is (kinda) the whole point of this course.

Definition 11.3 The *derivative of a function f at a number a* is

Wikipedia: derivative of a function f at a number a

11.1 The derivative

Recall that we've so far been looking at the derivative of a function f at a fixed number a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

What if we want to look at the derivative at an arbitrary point, not just at a? For this we replace the number a by a variable.

We already did this when we were looking at acceleration. Notice that this allows us to look at f' as a function! This function, f', is called the *derivative of f*.

Example 11.4 Let's do a test run with a super easy example. Find the derivative of $f(x) = x$.

Therefore $f'(x) = 1$.

Problem 11.5 Try a more complicated example with a partner. Find the derivative of $f(x) = x^2 - x$.

There are a lot of different ways that people write the derivative $f'(x)$. This comes from the history of derivation!

$$f'(x) = \dot{y} = \frac{dy}{dx} = \frac{df}{dx} = Df$$

Fractions are not ratios!!! You can't just divide

Definition 11.6 We say that a function f is *differentiable at a* if $f'(a)$ exists.

Wikipedia: differentiable at a

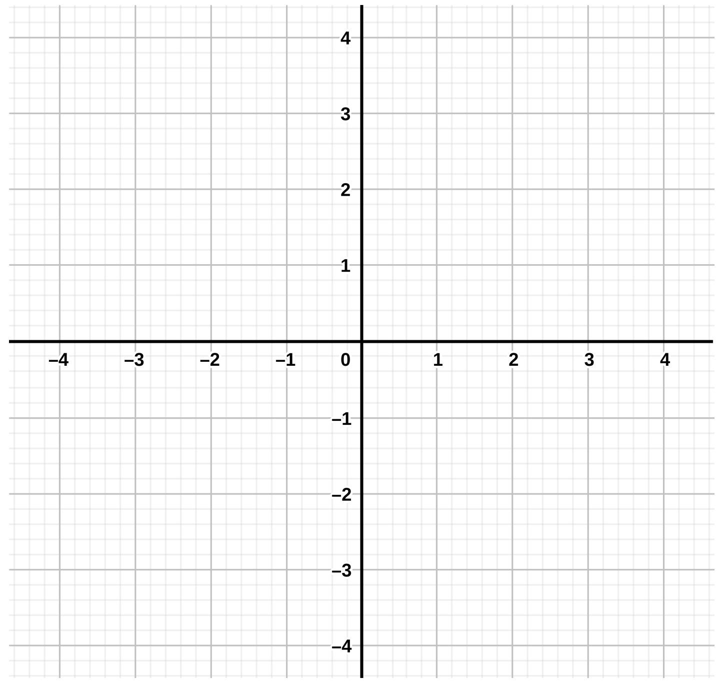
Example 11.7 Where is the following function differentiable?

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Being differntiable, actually tells us whether our function is continuous or not!

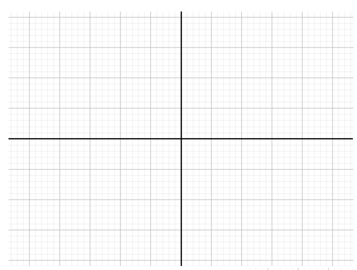
Theorem 11.8 *If f is differentiable at a, then f is continuous at a.*

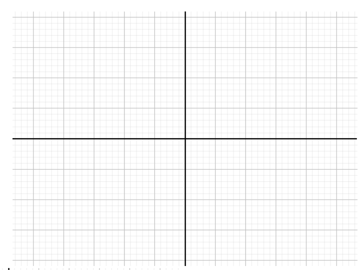
We see this already in our example!

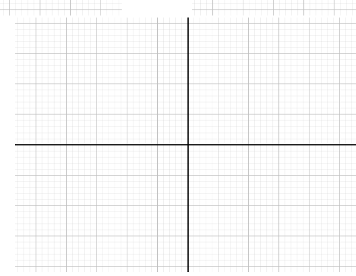


The opposite direction is *not* true! There are functions that are continuous at a point but not differentiable!

Intuitively, there are three main cases when a function is not differentiable:







11.2 Higher derivatives

Recall that when we were in outerspace that acceleration was calculated as the instantaneous rate of change (aka derivative) of the velocity. Velocity itself is also a rate of change! It is the instantaneous rate of change of position. So if velocity is the derivative of position, and acceleration is derivative of velocity then acceleration is the derivative of the derivative of position! There are two derivatives!

This happens because we're able to view the derivative as a function itself. And since it's a function, we can ask what is the derivative of the derivative. This is normally called the *second derivative*. In a similar fashion we can define the *third derivative*, *fourth derivative*, *n-th derivative*, etc. Notation-wise, this is what they look like:

$$
\begin{aligned}
2^{nd}: &\quad f''(x) = \ddot{y} = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = D^2f \\
3^{rd}: &\quad f'''(x) = \dddot{y} = \frac{d^3y}{dx^3} = \frac{d^3f}{dx^3} = D^3f \\
4^{th}: &\quad f''''(x) = f^{iv}(x) = f^{(4)}(x) = \ddddot{y} = \overset{4}{\dot{f}} = \frac{d^4y}{dx^4} = \frac{d^4f}{dx^4} = D^4f \\
n^{th}: &\quad f^{(n)}(x) = \overset{n}{\dot{f}} = \frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = D^nf
\end{aligned}
$$

Problem 11.9 With a partner, find the third derivative of $f(x) = x^2$.

12 Derivatives of various functions

12.1 Derivatives of polynomials

All of the above is fun, but this can get really complicated, really quickly. So instead, let's try and generalize our results so that we don't have to work as hard.

From our previous examples we see that:

$$\frac{d}{dx}(x^2) = \underline{\hspace{2cm}} \qquad \frac{d}{dx}(x) = \underline{\hspace{2cm}}$$

We're tempted to state the following (which turns out to be true!)

Theorem 12.1 *If n is a real number, then*

Proof. We're actually gonna prove this one for positive integers, because it's nice and (relatively) easy. Two parts:

(1) ____________________

(2)

□

Example 12.2

- What is the derivative of $f(x) = x^{93}$? ________.
- What is the derivative of $f(x) = \sqrt[5]{x^3}$? ________.
- What is the derivative of $f(x) = x^{\pi}$? ________.

Using our limit laws we have the following theorem:

Theorem 12.3 *Let f and g be differentiable functions and c a constant.*

- ____________________
- ____________________
- ____________________

This means we can differentiate more complicated polynomials!

Example 12.4 What is the derivative of $f(x) = x^9 + 3x^4 - x^2 + 1$?

Example 12.5 Let's apply this to something real world. Suppose we have a rod (or a piece of wire) of metal. The rod is "homogeneous" if the density of the rod is linear throughout. In other words it is defined to be the mass (kg) per unit length (m). We can write this as ρ (density) is equal to the mass m divided by length ℓ:

$$\rho = \frac{m}{\ell}$$

But, now suppose our rod is not homogeneous. Instead, suppose that the mass is given in some function $m = f(x)$. Then, the average density is:

$$\rho = \frac{\Delta m}{\Delta \ell} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This looks exactly like the derivative! So then the density at any given point is given by:

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta \ell} = \frac{dm}{dx}$$

For example, let $f(x) = x^{3/2}$. Using the limit rules we just saw, we know $f'(x) =$ __________. In other words at any point x on our rod, $\rho =$ __________.

So if we look 1 meter up our rod, then we know the density is:

12.2 Exponential functions

Let's talk about exponential functions and what their limits look like. Recall that an exponential function is a function of the form $f(x) = b^x$ for b a real number. Let's calculate its derivative.

$$\begin{aligned} f'(x) &= \underline{\hspace{10cm}} \\ &= \underline{\hspace{7cm}} \\ &= \underline{\hspace{7cm}} \\ &= \underline{\hspace{6.5cm}} \\ &= \underline{\hspace{8cm}} \\ &= b^x f'(0) \end{aligned}$$

But now we need to figure out what $f'(0)$ is! There is no easy way to do this, so we need to go back to our old methods of looking at multiple values of h.

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$
0.1	$0.7177346\ldots$	$1.1612317\ldots$
0.01	$0.695555\ldots$	$1.104669\ldots$
0.001	$0.693387\ldots$	$1.099216\ldots$
0.0001	$0.693171\ldots$	$1.0098673\ldots$

It turns out that

$$b = 2 \Rightarrow f'(0) = \lim_{x\to 0} \frac{2^x - 1}{x} = \underline{\hspace{3cm}}$$
$$b = 3 \Rightarrow f'(0) = \lim_{x\to 0} \frac{3^x - 1}{x} = \underline{\hspace{3cm}}$$

So, we have

$$\frac{d}{dx}2^x = \underline{\hspace{4cm}} \qquad \frac{d}{dx}3^x = \underline{\hspace{4cm}}$$

and in general:

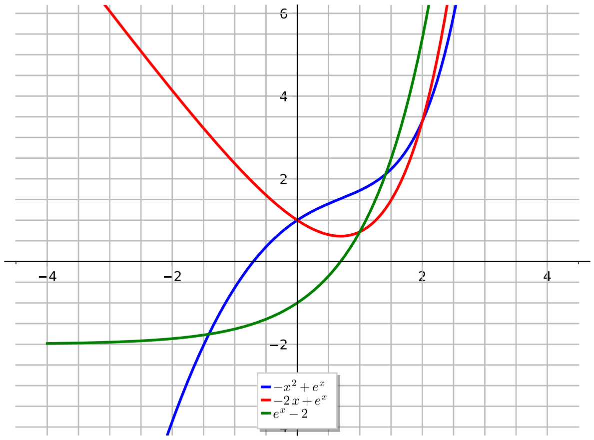
$$\frac{d}{dx}b^x = \underline{\hspace{4cm}}$$

What would be really cool is if we could find a number where $f'(0) = 1$. This would imply ____________________________________!! This is where the number __ comes from. If ________ then ____________________ and we are happy.

Example 12.6 Find the second derivative of $f(x) = e^x - x^2$.

$$f''(x) = e^x - 2$$

The graph of these functions is the following:

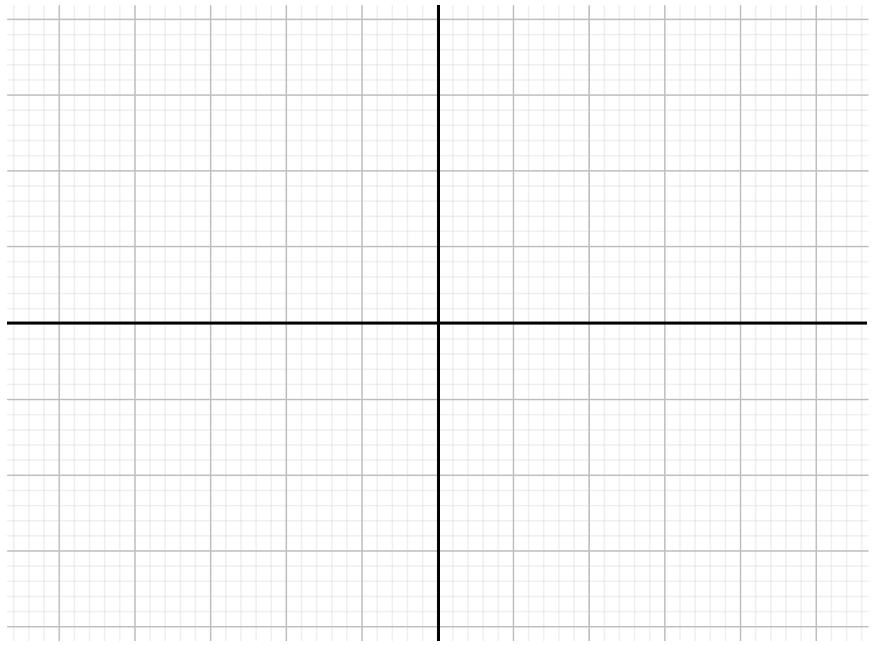


12.3 Derivatives of trig functions

Let's try and find the derivative of $\sin(x)$. By definition we have:

$$
\begin{aligned}
f(x) &= \sin(x) \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \underline{\hspace{10cm}} \\
&= \underline{\hspace{8cm}} \\
&= \underline{\hspace{7cm}} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}
\end{aligned}
$$

So we need to figure out what these last two limits are to find the derivative. We're going to use the squeeze theorem! First, we want to show that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$ where θ is an angle.



Therefore, intuitively, we know that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$.

Now, we take the limits of everything as θ goes to 0.

$$\lim_{\theta \to 0} \cos(\theta) = \underline{\hspace{6cm}}.$$

Therefore, by the squeeze theorem $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = \underline{\quad}$.

Now let's look at $\frac{\cos(\theta) - 1}{\theta}$.

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} &= \underline{\hspace{15cm}} \\
&= \underline{\hspace{20cm}} \\
&= \underline{\hspace{10cm}} \\
&= \underline{\hspace{14cm}} \\
&= -\lim_{\theta \to 0} \frac{\sin(\theta)}{\cos(\theta) + 1} \\
&= \underline{\hspace{4.5cm}} \\
&= \underline{\quad}
\end{aligned}$$

Therefore,

$$f'(x) = \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} = \underline{\hspace{3.5cm}}$$

Using very similar methods, we can prove derivatives for all of the trig functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin(x)$	__________	$\csc(x)$	__________
$\cos(x)$	__________	$\sec(x)$	__________
$\tan(x)$	__________	$\cot(x)$	__________

13 Ways to find more complex derivatives

13.1 Product and quotient rules

Recall that when we did addition and subtraction of derivatives, we saw that a very similar thing happened as with the limit laws; we could just separate them. For the limit laws we had:

$$\lim_{x\to a}(f(x)+g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$$

And for derivatives we had:

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

We now ask the same question for multiplication since we have the same limit law:

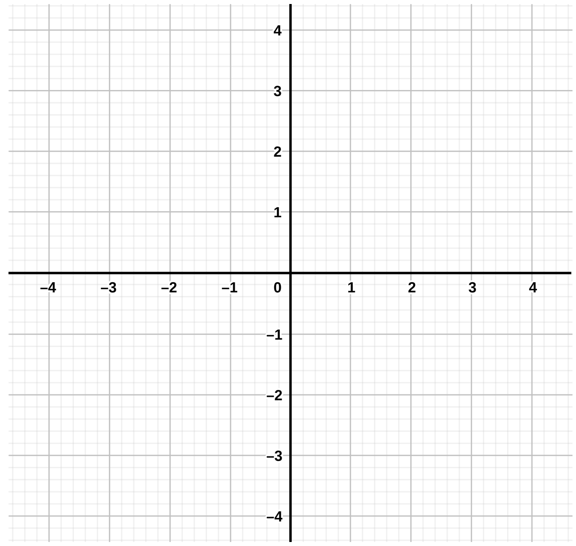
$$\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$

It turns out the same property does *not* hold for multiplication (and division)! In particular we have $(fg)' \neq f'g'$ like we would want.

Example 13.1 Let's look at a super simple example for proof. Let $f(x) = x$ and $g(x) = x$. Then,

So we need to figure something else out to calculate $(fg)'$. This is a bit complicated, but it turns out the best way is to use geometry.

First, let's look at one of our functions.


4
3
2
1
–4 –3 –2 –1 0 1 2 3 4
–1
–2
–3
–4

This is called the *product rule* : If f and g are both differentiable, then

Wikipedia: product rule

Example 13.2 Let's do an example. Let $f(x) = x^2e^x$. What is $f'(x)$?

We can also do quotients based off a similar technique as the product.

Wikipedia: quotient rule

This is called the *quotient rule* : If f and g are both differentiable and $g(x) \neq 0$, then

Note that in the quotient rule, the *order* matters since we are subtracting. Make sure to keep this in mind.

Example 13.3 Let $f(x) = \frac{x^3+3x}{x^2+1}$.

Therefore,

$$f'(x) = \frac{x^4+3}{x^4+2x^2+1}$$

Problem 13.4 With a partner find $f'(x)$ where $f(x) = \frac{x^4+3x^2+x-1}{e^x}$.

Example 13.5 We can also use the quotient rule to find the derivative of $\tan(x)$. Suppose that $f(x) = \frac{\sin(x)}{\cos(x)} = \tan(x)$ and find $f'(x)$.

Problem 13.6 With a partner, find the derivative of $f(x) = \sin(x)\cos(x)$.

13.2 The chain rule

We did addition, subtraction, multiplication and division, but there's one operation we haven't touched yet: composition. What is the derivative of $(f \circ g)$?

In prime notation, we have

Wikipedia: chain rule

This is called the *chain rule* .

Example 13.7 Let $f(x) = \sqrt{\sin(x)}$. What is $f'(x)$?

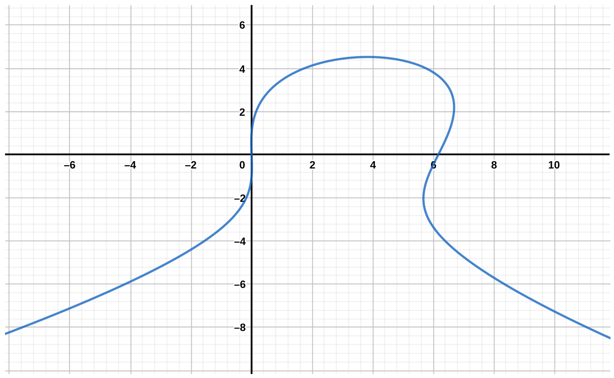
Problem 13.8 Let $f(x) = \frac{1}{\sqrt[3]{x^2+3x}}$. With a partner, find $f'(x)$.

Let's use the chain rule to find the derivative of an exponential function. Let $f(x) = b^x$, and we want to find $f'(x)$. We're going to use the fact that $b = e^{\ln(b)}$. So we have

13.3 Implicit differentiation

So far, we've been looking at functions that have been explicitely defined: $f(x) = \cdots$. But what happens if our function is implicitly defined:

$$y^3 + 6x^2 - 2xy = 37x + y$$



Are there ways to find the tangent to the curve? (aka the derivative)

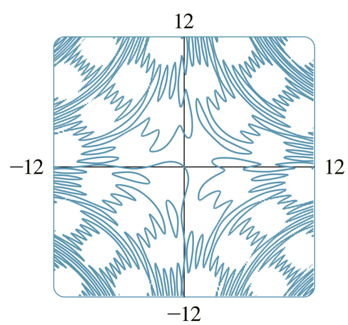
What we do is we break down our differentiation by small sections. This is best done with an example.

Example 13.9 Let $x^3 + y^3 = 37$.

Let's try a little more complicated example.

Example 13.10 Let $y^3 + 6x^2 - 2xy = 37x + y$.

Problem 13.11 Let $\sin(xy) = \sin(x) + \sin(y)$.



With a partner, find y'.

Example 13.12 Let $x^3 + y^3 = 37$. Find y''.

14 Some more derivatives

14.1 Derivatives of inverse trig functions

We can use implicit differentiation in order to find the derivatives of inverse trig functions. Recall that $\arcsin(x) = \sin^{-1}(x) = y$ means that $\sin(y) = x$ when $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$. So let's test it out.

$$\begin{aligned}
\frac{d}{dx}\sin(y) &= \frac{d}{dx}x \\
\frac{d}{dy}\sin(y)\frac{dy}{dx} &= 1 \\
\cos(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos(y)}
\end{aligned}$$

Since $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ then $\cos(y) \geq 0$. And since $\sin^2(y) + \cos^2(y) = 1$ we have

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \underline{\hspace{4cm}}.$$

In other words,

Using similar approaches, we can find the derivatives for every inverse trig function:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\arcsin(x)$	____________	$\text{arccsc}(x)$	________________
$\arccos(x)$	________________	$\text{arcsec}(x)$	____________
$\arctan(x)$	_________	$\text{arccot}(x)$	____________

14.2 Derivatives of logarithm functions

What is the derivative of the function $f(x) = \log_b(x)$?

Recall that $y = \log_b(x)$ means that $b^y = x$. So let's implicit differentiate!

$$\frac{d}{dx}b^y = \frac{d}{dx}x$$

$$\underline{\qquad\qquad\qquad} = \_\_$$

$$\underline{\qquad\qquad\qquad\qquad} = \_\_$$

$$\frac{dy}{dx} = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

The derivative comes from Problem 13.2.

What this means is that if $b = e$ then

Example 14.1 Let $f(x) = \ln(\cos(x))$, what is $f'(x)$?

Example 14.2 Let $f(x) = \left(\ln\left(\frac{x^3}{x^2+1}\right)\right)^2$, what is $f'(x)$?

Problem 14.3 Let $f(x) = x^2 \ln\left(\frac{x+3}{x^2}\right)$, with a partner, find $f'(x)$.

The logarithm can actually help us differentiate more complicated rational functions. This is best demonstrated with an example.

Example 14.4 Let

$$f(x) = \frac{x^{\frac{9}{4}}(x^3-1)^{\frac{4}{3}}}{(7x^2-1)^4}$$

and find $f'(x)$. This equation is super complicated and we can use the chain rule, the product rule and the quotient rule to solve it, but that's going to get very complicated very fast. Instead, we can use logarithms to simplify everything. We do this in 3 steps.

(1) ____

(2) ____

(3) ____________________

Example 14.5 Let's try it again with a super simple example. Let $f(x) = x^x$.

(1)

(2)

(3)

Problem 14.6 Let

$$f(x) = \frac{x^{\frac{4}{3}}\left(x^3 - 3x^2\right)^{\frac{7}{3}}}{\sqrt{x^2+1}}$$

with a partner, find $f'(x)$.

15 Uses of derivatives

15.1 Maximum and minimum values

A lot of problems in life are optimization problems: when do functions reach maximum/minimum values? We kinda saw this with our astronaut problem where we wanted to find maximum acceleration. We also see this for example when trying to decide:

- What was the peak value of CO_2 in the atmosphere before 0 BCE?
- What time do restaurants have the most amount of clientele?
- What shape of packaging for our product do we need to minimize costs?

There are an endless supply of max/min problems.

Problem 15.1 Try and come up with a max/min problem and share it with your neighbor:

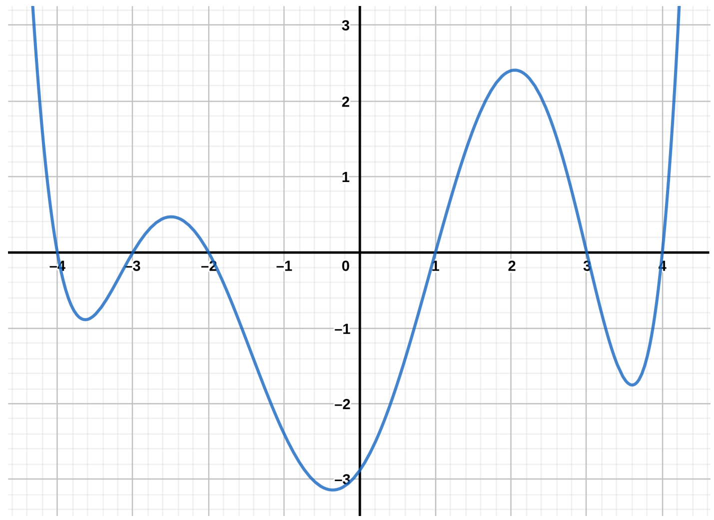
Max/Min problem: ____

____

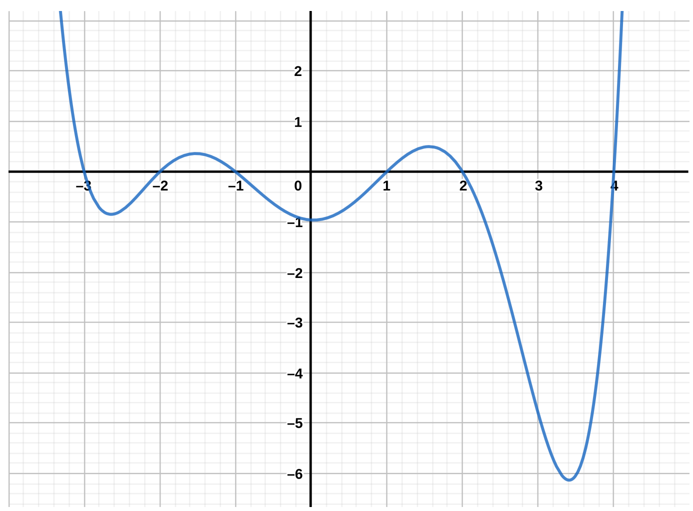
Mathematically, what is maximum and minimum though? A function f has an *absolute maximum value at c on a domain D* if $f(c) \geq f(x)$ for every $x \in D$. Similarly, a function f has an *absolute minimum value at c on a domain d* if $f(c) \leq f(x)$ for every $x \in D$. Absolute values are sometimes known as global values. An *extreme value of f* is an ________ ____ of f.

We can also look at points which are maximal/minimal in a smaller area. If $f(c) \geq f(x)$ for every x near c then we say $f(c)$ is a *local maximum*. Similarly, $f(c)$ is a *local minimum* if $f(c) \leq f(x)$ for every x near c.

Example 15.2



Problem 15.3 With a partner, fill in the absolute max/min and the local max/mins in the domain $[-3, 4]$ on the following graph:



How do we know that extreme values even exist? Maybe we have some graph where there are no extreme values! This turns out not to be the case as was shown by Bernard Bolzano in the 1830s when he proved the following result.

Theorem 15.4 (Extreme value theorem) *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

What about local maxima and minima? It turns out that Pierre Fermat gave a nice property on local maxima and minima.

Theorem 15.5 (Fermat's Theorem) *If f has a local maximum or minimum at c and if $f'(c)$ exists then $f'(c) = 0$.*

Be very careful with this theorem! It's very easy to want to say that the theorem says more than it actually does. There are 3 common errors that students make with this theorem, two of which are mentioned in the book.

(1) ____

(2) ____

(3) ____

Although there are a lot of easy pitfalls with Fermat's theorem, it does suggest that *maybe*, we can determine how a function looks based on its derivative. A *critical number* of a function f is a number c (in the domain) such that $f'(c) = 0$ or $f'(c)$ does not exist. In other words, Fermat's theorem becomes:

Wikipedia: critical number

If f has a local maximum or minimum at c then c is a critical number of f.

It turns out we can use cricital numbers to tell us when we have an absolute max or min in some interval $[a, b]$. For this we:

(1) Find the critical values of f in the interval (a, b).

(2) Find the values of f at the critical values and at the end points of the interval (at a and b).

(3) The largest value of the previous step is the absolute maximum and the smallest value is the absolute minimum.

Example 15.6 Let $f(x) = x^3 - x^2 - x$, find its absolute maximum and minimum on the interval $[\frac{-1}{2}, 2]$.

Problem 15.7 Let $f(x) = x^3 + x^2 - 1$, find its extreme values on the interval $[-1, 1]$.

15.2 The mean value theorem

Our next goal is the mean value theorem. We start off with Rolle's theorem, which was named after Michel Rolle who published it in 1691 for polynomial functions. Although Rolle was the first to publish his ideas, Indian mathematician Bhāskara II (1114–1185) is credited to be the first person we know of to have knowledge of the theorem. In addition, the theorem itself was not proved in full until 1823 when Cauchy proved it in its entirety.

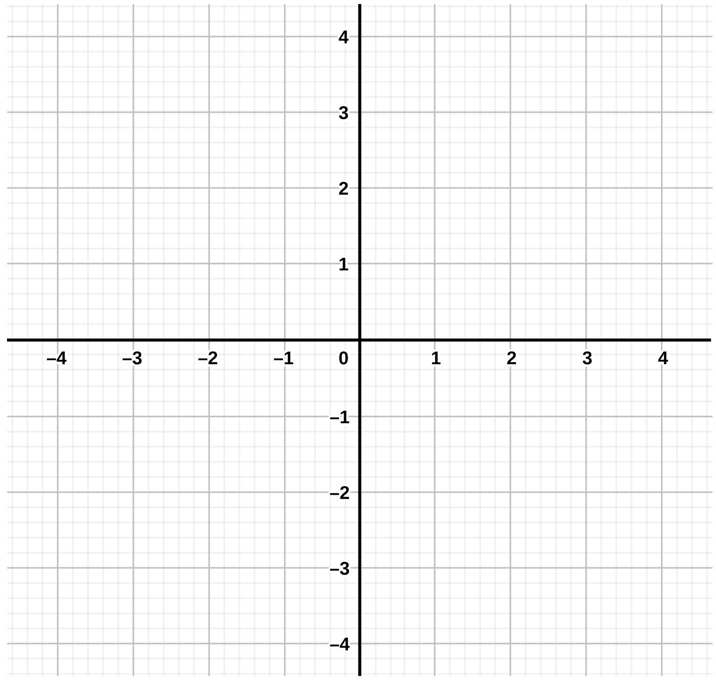
Theorem 15.8 (Rolle's theorem) *Let f be a function that satisfies the following three hypotheses:*

(1) f is continuous on $[a, b]$.

(2) f is differentiable on (a, b).

(3) $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.



Rolle's theorem is used to find the number of roots a function might have.

Example 15.9 Show that the equation $f(x) = x^3$ has exactly one real root.

The mean value theorem was known first to Parameshvara (1370–1460), an Indian mathematician. In Europe, it was first stated by Joseph-Louis Lagrange in the following form.

Theorem 15.10 (Mean value theorem) *Let f be a function such that:*

(1) f is continuous on $[a, b]$.

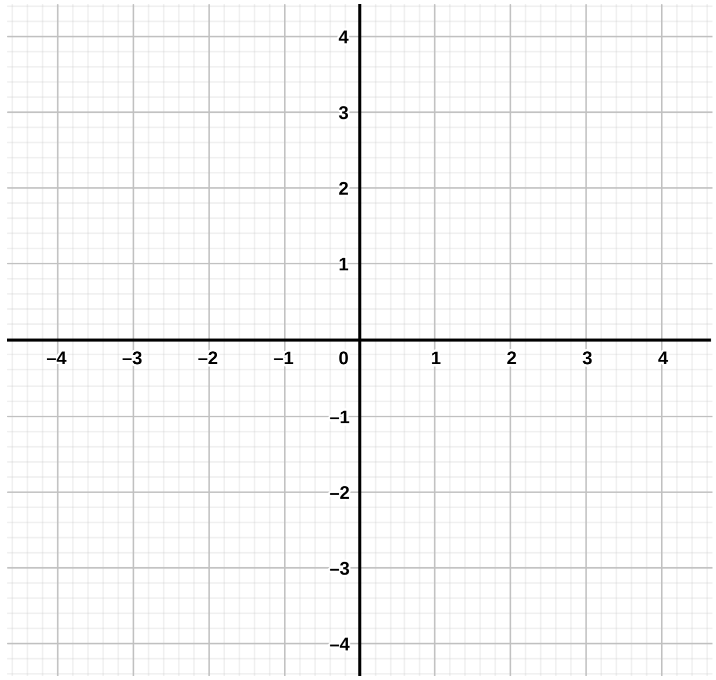
(2) f is differentiable on (a, b).

Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Sometimes this theorem is known as *Lagrange's mean value theorem* . There is an extended version of this theorem which we won't go over called the *extended mean value theorem* or *Cauchy's mean value theorem*.

Wikipedia: Lagrange's mean value theorem



Example 15.11 Let $f(x) = x^2 - 2$ and let's look at the interval $[0, 2]$. Show the mean value thorem is true on this interval.

The mean value theorem allows us to state the following two theorems.

Theorem 15.12 *If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).*

Corollary 15.13 *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); i.e., $f(x) = g(x) + c$ for some constant c.*

16 L'Hôpital's Rule

We're going to go back to limits for a minute and look at how we can find limits at places that aren't that well-defined. For example, what's the following limit:

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \underline{\qquad}$$

Or how about

$$\lim_{x\to \infty}\frac{x^2}{e^x} = \underline{\qquad\quad}$$

Whenever we have a limit

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \underline{\qquad}$$

then this limit is called an *indeterminate form* of type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

These limits aren't very easy to solve by the methods we have so far, but in 1694 the Swiss mathematician Johann Bernoulli introduced a theorem to the French mathematician Guillaume de l'Hôpital which we now call L'hôpital's rule:

Theorem 16.1 (L'Hôpital's rule) *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose further that either*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

(in other words, the limit is an indeterminate form of either type). Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \underline{\qquad\qquad\qquad\qquad}$$

if the limit on the right side exists or is $\pm\infty$.

This is awesome! Let's try it with our examples from before.

Example 16.2 Let $f(x) = \sin(x)$ and $g(x) = x$ and, from above, recall

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \underline{\qquad}$$

so we can use L'hôpitals rule!

Example 16.3 Let $f(x) = x^2$ and $g(x) = e^x$ and, from above, recall that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

We can therefore use L'Hôpital's rule!

Try another example with a partner.

Problem 16.4 Let $f(x) = x^2 - x - 2$ and $g(x) = \sqrt{2} - \sqrt{x}$. With a partner, find

$$\lim_{x\to 2} \frac{f(x)}{g(x)}$$

Example 16.5 Which of the following can I use L'hôpital's rule on?

$$\lim_{x\to\infty} \frac{1}{x} \qquad \lim_{x\to\infty} \frac{x^2+7x}{x^2+9x+3} \qquad \lim_{x\to\infty} \frac{x^2+3x+1}{(x^4+7x)\cdot e^x}$$

$$\lim_{x\to 0} \frac{e^x}{x} \qquad \lim_{x\to\pi} \frac{\pi - x}{\cos(x)} \qquad \lim_{x\to 0} \frac{0}{\cos(x)}$$

We can also use L'Hôpital's rule in other cases! In most of these cases, what we want to do is convert our limit into a limit of one of our two indeterminate types. These are best done with examples.

16.1 The form $\infty - \infty$

Example 16.6 Find the limit

$$\lim_{x\to 0} \frac{1}{x\sin(x)} - \frac{1}{x^2}$$

$$\lim_{x\to 0} \tfrac{1}{x\sin(x)} - \tfrac{1}{x^2} \quad =$$

$$= \tfrac{1}{6}$$

Notice how we changed our formula from $\infty - \infty$ to $\frac{0}{0}$ in order to be able to use L'Hôpital's rule. We do the same for all the types.

16.2 The form $0 \cdot \pm\infty$

Example 16.7 Find the limit

$$\lim_{x\to 0} 8x^4 \ln(3x)$$

$$\lim_{x\to 0} 8x^4 \ln(3x) \quad =$$

$$= 0$$

16.3 The forms 0^0, ∞^0, and $1^{\pm\infty}$

We can do the same with powers! In these cases, it's a little more complicated.

Example 16.8 Find the limit

$$\lim_{x \to 0} (x^2)^{x^2}$$

$$\lim_{x\to 0} (x^2)^{x^2} \quad \to$$

$$\lim_{x\to 0} (x^2)^{x^2} \quad = e^0 = 1$$

In essence, we followed the following steps:

(1) ______________________________

(2) ______________________________

(3) ______________________________

Problem 16.9 With a partner, find the following limit.

$$\lim_{x \to 0} \sin(x)^x$$

Chapter 5

Integration

17 Antiderivatives

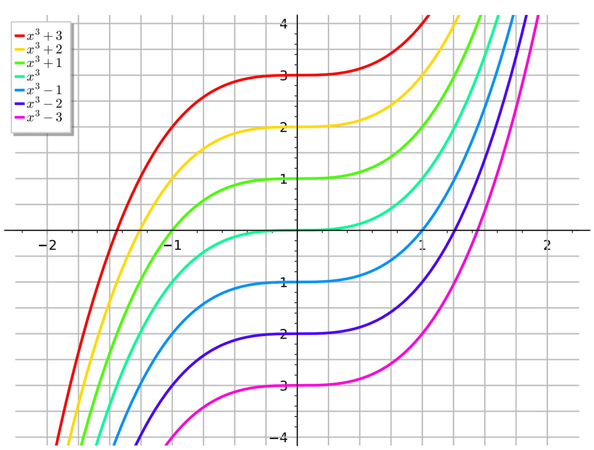
We've been looking at how to find the derivative of a function $f(x)$. But what happens if we want to go backwards? Say we have some function $f(x)$ and we want to know whose derviative it is. This is the idea of antiderivatives.

Definition 17.1 A function F is called an *antiderivative* of f on an interval I if ______________________ for all x in I.

Wikipedia: antiderivative

Example 17.2 If $f(x) = 3x^2$, what is its antiderivative? __________

What do these look like?



But how do we know that's all? What if there is some other antiderivative? This turns out not to be the case because of the following theorem.

Theorem 17.3 *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

Example 17.4 What are the (most general) antiderivatives of the following functions:

(1) $f(x) = \cos(x)$: ________________

(2) $f(x) = x^n$ where $n \neq -1$: ________________

(3) $f(x) = x^{-1}$: ________________

We can use our derivative laws to help us find the antiderivatives of more complicated functions. For example, if we have $f(x) + g(x)$ and if $F(x)$ and $G(x)$ are the antiderivatives of f and g respectively, then $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$. Let's see an example.

Example 17.5 Find all antiderivatives of $f(x) = \frac{1}{\sqrt{1-x^2}} - 2\cos(x) + \frac{2x^5 + \sqrt[3]{x}}{x^2}$.

Problem 17.6 Try one with a partner. Let $f(x) = 2\sin(x) - \frac{1+x^3}{x}$ and find all antiderivatives of $f(x)$.

If we're looking for a particular answer, we also give a data point with our function.

Example 17.7 Find f if $f'(x) = e^x - 2\cos(x)$ and $f(0) = 2$.

Let's try something a little harder.

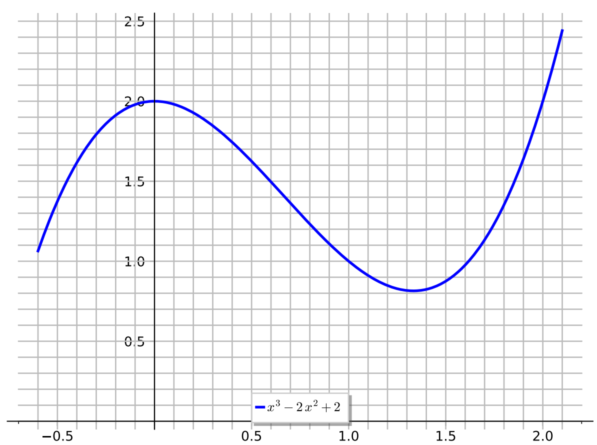
Example 17.8 Find f if $f''(x) = 3x+2$ where $f(0) = 4$ and $f(2) = 13$.

Problem 17.9 With a partner, find f if $f''(x) = 4x^2 + 1$ where $f(0) = \frac{1}{6}$ and $f(1) = 2$.

Function	Derivative	Function	Derivative
$f(x) = x^r$	$f'(x) = rx^{r-1}$	$f(x) = kg(x)$	$f'(x) = kg'(x)$
$f(x) = g(x)h(x)$		$f(x) = \frac{g(x)}{h(x)}$	$f'(x) = \frac{g'(x)h(}{}$
$f(x) = g(h(x))$		$f(x) = g(x)^r$	$f'(x) = r(g(x)$
$f(x) = e^x$	$f'(x) = e^x$	$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$
$f(x) = a^x$	$f'(x) = a^x \ln(a)$	$f(x) = \log_a(x)$	$f'(x) = \frac{1}{x\ln(a)}$
$f(x) = \sin(x)$		$f(x) = \cos(x)$	
$f(x) = \tan(x)$	$f'(x) = \sec^2(x)$	$f(x) = \cot(x)$	$f'(x) = -\csc^2$
$f(x) = \sec(x)$	$f'(x) = \sec(x)\tan(x)$	$f(x) = \csc(x)$	$f'(x) = -\csc($
$f(x) = \arcsin(x)$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$	$f(x) = \arccos(x)$	$f'(x) = \frac{-1}{\sqrt{1-x^2}}$
$f(x) = \arctan(x)$	$f'(x) = \frac{1}{1+x^2}$	$f(x) = \operatorname{arccot}(x)$	$f'(x) = \frac{-1}{1+x^2}$
$f(x) = \operatorname{arcsec}(x)$	$f'(x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$f(x) = \operatorname{arccsc}(x)$	$f'(x) = \frac{-1}{\lvert x\rvert\sqrt{x^2}}$

18 Areas and distances

Suppose we have the following function: $f(x) = x^3 - 2\,x^2 + 2$:



Since we already kinda know how to look at tangents to a curve, let's look at area under a curve. Say we want to calculate the area under the curve between 0 and 2. How can we do this?

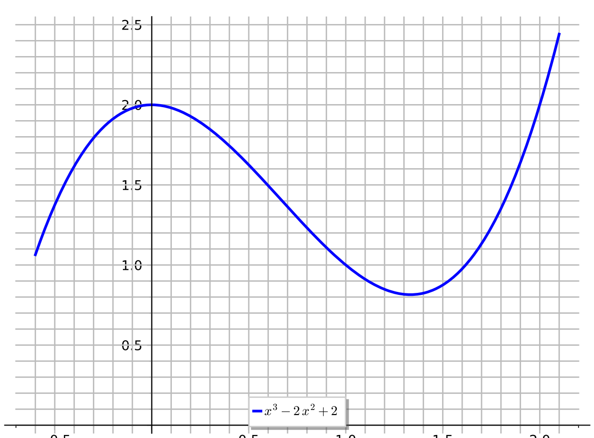
Using our techniques, we can estimate that the area should be: ___ .

In essence, we are generally using the following formula for trying to calculate the area:

$$\sum_{i=0}^{n} f(x_i)\Delta x$$

But we can look at this function differently. We can get an estimation by looking at the right-hand sides of our rectangles:

$$\sum_{i=0}^{n} f(x_{i+1})\Delta x$$



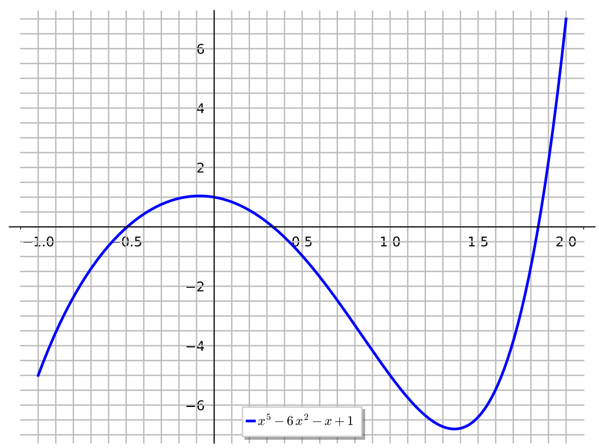
Doing this with our example, we'd get:

$$\begin{aligned}
&\sum_{i=0}^{9} f\left(\frac{i+1}{5}\right)\Delta x \\
&\quad = \left(\frac{241}{125}+\frac{218}{125}+\frac{187}{125}+\frac{154}{125}+1+\frac{106}{125}+\frac{103}{125}+\frac{122}{125}+\frac{169}{125}+2\right)\frac{1}{5} \\
&\quad = \frac{67}{25} \approx 2.68000000000000
\end{aligned}$$

Notice how these two values are similar to one another!

19 Definite intgral

Now let's look at a different function, and we're going to be a little more precise in how we are defining these rectangles. We will work with $f(x) = x^5 - 6\,x^2 - x + 1$:



Given a closed interval $[a,b]$, a *partition* of an interval of numbers x_i is such that:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

This gives us little sub-intervals:

$$[x_0, x_1],\ [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

If the intervals are all the same size $(\frac{b-a}{n})$ we say that the partition is *regular*.

Example 19.1 For some examples, let $a = 2$ and $b = 5$. The following are two different partitions:

$$2 < 2.1 < 3 < 4.2 < 4.5 < 4.51 < 5$$
$$2 < 3 < 4 < 5$$

and they each define different intervals:

$$[2, 2.1],\ [2.1, 3],\ [3, 4.2],\ [4.2, 4.5],\ [4.5, 4.51],\ [4.51, 5]$$
$$[2, 3],\ [3, 4],\ [4, 5]$$

Which partition is regular? ______________

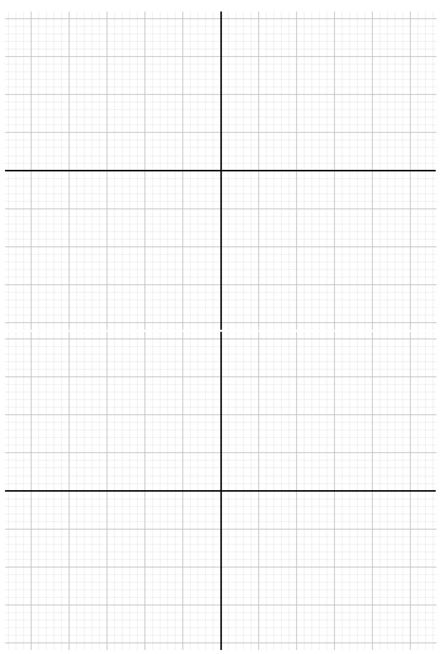
Problem 19.2 Given $a = 0$ and $b = 1$, with a partner, give a regular partition:

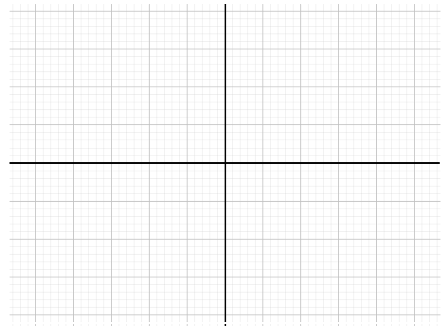
Always assume regular. The *length of an interval* is the difference between the start and end of the interval: $\Delta x_i =$ ________________ for the interval $[x_{i-1}, x_i]$.

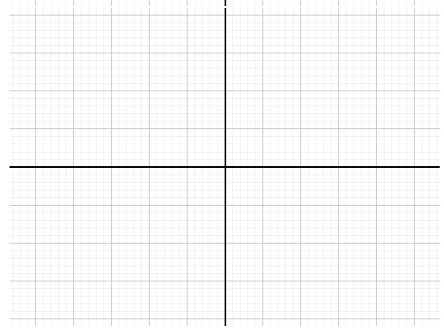
We are now ready to define our summations! The *Riemann sum* of a function on an interval $[a, b]$ is the sum:

Wikipedia: Riemann sum

where $x_i^\star \in [x_{i-1}, x_i]$ is some arbitrary number.







We've already seen two of these! The *right-hand sum* is the Riemann sum where we let $x_i^\star = x_i$. The *left-hand sum* is the Riemann sum where we let $x_i^\star = x_{i-1}$. The *midpoint sum* is the Riemann sum where we let $x_i^\star = \frac{x_{i-1}+x_i}{2}$.

But what we reeeeaaaallllly want is to find the *exact* area, not just approximations! What can we do to find the exact area?

We're going to use our favorite tool so far in class: limits. Yes... limits.

The limit we're going to use is actually the following limit:

If this limit exists we call say that the function is *Riemann-integrable*.

BUT, this notation is horrible. There are a million and a half symbols. Who wants to write that each time? No one. So we're going to use another notation that (I think) everyone has already seen: the integral.

Remark 19.3 Some history! We use Σ for summation because summation starts with s. So, we need a new symbol for the integral. Luckily, Gottfried Leibniz (German) had came up with the perfect notation back in 1675! He used ancient German's long s: ſ. (Nowadays, German's long s is written as "ß".) Why? Because the integral is a summation of realy small intervals. So ſ is perfect for that since it keeps it as an "s", but is a different s! (Even in English we used to have ſ! It wasn't until between 1800 and 1820 that this letter disappeared from English.)

We changed the "s", but we also need to change the Δ since we're taking the limit. Since we're taking the limit, these intervals are getting smaller and smaller. In other words, it would be nice to represent that by using a "small" Δ. So what's the lower-case of Δ? It's δ of course! But, Leibniz, who was one of the founders of calculus, translated the greek to German and the δ became a d. This is where Leibniz notation: $\frac{dy}{dx}$ actually came from!

So, *if* our function is Riemann-integrable, we will write:

$$\underline{\hspace{4cm}} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x_i.$$

Some definitions:

- The a is the *lower limit of integration*,
- The b is the *upper limit of integration*,
- The $f(x)$ is the *integrand*,
- The x is called the *variable of integration* or the *independent variable*.

Theorem 19.4 *If a function $f(x)$ is __________________ on the interval $[a,b]$ or if $f(x)$ has only a finite number of jump discontinuities, then the function $f(x)$ is ____________________________________ on $[a,b]$.*

Example 19.5 Let's look at an (easy) example. Find $\int_1^2 x^2\, dx$.

That was disgusting. The whole point of the next few days is to try and make this easier.

19.1 Properties of definite integrals

Proposition 19.6 *Let $f(x)$ be a continuous function on $[a, b]$ and let c be a constant.*

(1) $\int_a^b f(x)\,dx = \int_a^b f(t)dt = \int_a^b f(u)\,du$

(2) $\int_a^a f(x)\,dx =$ __

(3) $\int_a^b c\,dx = c(b-a)$

(4) $\int_a^b f(x)\,dx =$ ____________________

(5) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(6) $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(7) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$ ________________

19.2 The fundamental theorem of calculus

Remember that in summations we had one summation which we called the telescoping sum:

$$\sum_{i=1}^{n} a_i - a_{i-1} = \underline{\hspace{3cm}}.$$

In fact, our integral is written in almost the same way:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^\star)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^\star)(x_i - x_{i-1}).$$

So we now want to know if there exists an easier way to solve these integrals and we will see (very soon) that a way exists! In fact, this is done through antiderivatives!

Recall that an antiderivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Theorem 19.7 (The fundamental theorem of calculus) *If $f(x)$ is a continuous function on an interval $[a, b]$ and if $F(x)$ is an antiderivative of $f(x)$ then*

$$\int_a^b f(x)\,dx = \underline{\hspace{6cm}}$$

To make life easier, here are some functions with some antiderivatives

Function	**(an) antiderivative**
x^n where $n \neq -1$	$\frac{x^{n+1}}{n+1}$
x^{-1}	
$\sin(kx)$ where $k \neq 0$	$-\frac{\cos(kx)}{k}$
$\cos(kx)$ where $k \neq 0$	$\frac{\sin(kx)}{k}$
$\sec^2(kx)$ where $k \neq 0$	
$\operatorname{cosec}^2(kx)$ where $k \neq 0$	$-\frac{\operatorname{cotan}(kx)}{k}$
$\sec(kx)\tan(kx)$ where $k \neq 0$	$\frac{\sec(kx)}{k}$
$\operatorname{cosec}(kx)\operatorname{cotan}(kx)$ where $k \neq 0$	$-\frac{\operatorname{cosec}(kx)}{k}$
e^{kx} where $k \neq 0$	

Example 19.8 Let's look at some examples:

$$\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8-1}{3} = \frac{7}{3}$$

$$\int_2^4 2x\,dx = \underline{\hspace{10cm}}$$

$$\int_0^\pi \sin(x)\,dx = \underline{\hspace{4cm}}$$

$$\int_5^5 e^{x^2} \cdot \sin^2(x^2)\,dx = \underline{\hspace{0.5cm}}$$

Problem 19.9 Try an example with a partner. Solve:

$$\int_1^3 4x^3 - e^x + \frac{1}{x}\,dx$$

Hint: Recall the properties of integrals from before.

20 Indefinite integrals

Because of the fundamental theorem of calculus, we can find the integral $\int_a^b f(x)\,dx$. But in order to find this, we must first find an antiderivative of $f(x)$!

This is not always easy!

Example 20.1 Find a function $F(x)$ such that $F'(x) = f(x) = e^{x^2}$.

For this, we need to find a way to generate antiderivatives. This is normally done by using indefinite integrals.

A *indefinite integral* is the set of ____ antiderivatives of a given function. In other words, it's the general formula $F(x)+c$ that we saw a few days ago. We will normally denote this by:

All this is saying is that $F(x) + c$ is the general formula for an antiderivative of $f(x)$.

Definite and indefinite integrals are different!!! Definite integrals have lower/upper limits and give you a value. Indefinite integrals have no lower/upper limits and give you a function.

We already have a ton of indefinite integrals that we can create using the derivative rules we saw earlier:

(1) $\int k\,dx = kx + c$

(2) $\int [k_1 f(x) + k_2 g(x)]\,dx = k_1 \int f(x)\,dx + k_2 \int g(x)\,dx$

(3) $\int x^n\,dx =$______________

(4) $\int \frac{1}{x}\, dx = \ln(|x|) + c$

(5) $\int e^x\, dx =$ __________

(6) $\int a^x\, dx = \frac{a^x}{\ln a} + c$ (where $1 \neq a > 0$)

(7) $\int \sin(x)\, dx =$ ____________________

(8) $\int \cos(x)\, dx =$ ________________

(9) $\int \sec(x)\, dx = \ln(|\sec(x) + \tan(x)|) + c$

(10) $\int \operatorname{cosec}(x)\, dx = \ln(|\operatorname{cosec}(x) - \operatorname{cotan}(x)|) + c$

(11) $\int \tan(x)\, dx = \ln(|\sec(x)|) + c$

(12) $\int \operatorname{cotan}(x)\, dx = \ln(|\sin(x)|) + c$

(13) $\int \sec^2(x)\, dx = \tan(x) + c$

(14) $\int \operatorname{cosec}^2(x)\, dx = -\operatorname{cotan}(x) + c$

(15) $\int \sec(x)\tan(x)\, dx = \sec(x) + c$

(16) $\int \operatorname{cosec}(x)\operatorname{cotan}(x)\, dx = -\operatorname{cosec}(x) + c$

(17) $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x) + c$

(18) $\int \frac{1}{x\sqrt{x^2-1}}\, dx = \operatorname{arcsec}(|x|) + c$

(19) $\int \frac{1}{1+x^2}\, dx = \arctan(x) + c$

But who cares? We care because it gives us an easier way to find the definite integral when we want to find it.

Example 20.2 Evaluate the following expression.

$$\int_2^4 \frac{12}{x^3}\, dx =$$

$$= \frac{9}{8}.$$

Problem 20.3 Try and find the following limit with a partner.

$$\int_1^4 \frac{1}{\sqrt{x^3}}\,dx$$

Example 20.4 Let's try a more complicated example. Let's find $\int_0^\pi 3x^2 - e^x + \cos(x)\,dx$.

$\int 3x^2 - e^x + \cos(x)\,dx =$ ______________________________

Therefore,

$$\int_0^\pi 3x^2 - e^x + \cos(x)\,dx =$$

$$= \pi^3 - e^\pi + 1$$

21 Solving integrals

21.1 Substitution

We're now going to look at the chain rule, but for integration. So suppose we have some (indefinite) integral like: $\int (3x+2)^{7/9}\,dx$? And we're trying to find the antiderivative of $(3x+2)^{7/9}$.

So far we haven't really learned how to tackle something like this, but we can use the ideas of the chain rule (and implicit differentiation) to solve something like this. What this method is called is "substitution".

What we want is an integral that is easier to handle. In our case, we want to change our integral to be something like $\int u^{7/9}\,du$. Why? Because we already know how to do this function!

So if we set $u = 3x+2$, then we have $\int u^{7/9}\,dx$. But now we have a different problem. What do you think the problem is?

Well, let's see if we can figure it out! We're gonna use implicit differentiation:

$$u = 3x + 2 \Rightarrow \underline{\hspace{6cm}}$$

Therefore

$$\begin{aligned}
\int (3x+2)^{7/9}\,dx &= \underline{\hspace{6cm}} \\
&= \frac{1}{3}\int u^{7/9}\,du \\
&= \underline{\hspace{6cm}} \\
&= \frac{9}{3\cdot 16}u^{16/9} + c \\
&= \frac{3}{16}(3x+2)^{16/9} + c
\end{aligned}$$

And done!

So what did we do?

(1) ______________________

(2) ____

(3) _____________________________________

(4) ____________________________

(5) ____

Problem 21.1 With a partner, find the following integral: $\int x(7x^2 - 8)^{16/7}\,dx$. **Hint:** Don't worry about the x by itself.

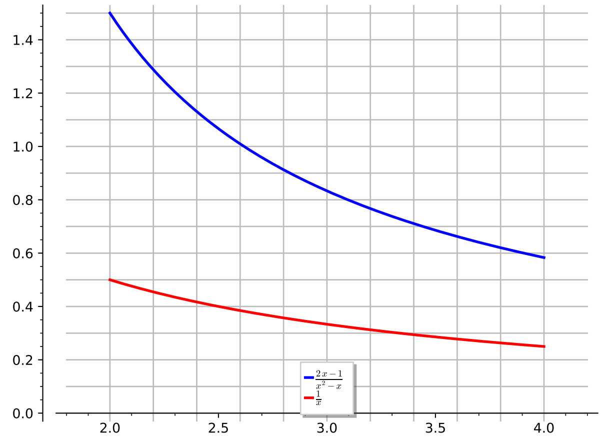
But recall that we're doing all of these things to find definite integrals! So what happens with that?

Example 21.2 Find the following definite integral: $\int_2^4 \frac{2x-1}{x^2-x}\,dx$

Therefore,

But now we have a different problem! ____

Let's look at the two functions together. The one on top (blue) is the one with variable x and the one on the bottom (red) is the one with variable u.



So how do we solve this? We do exactly like we did with dx.

Here's the substitution rule in all its glory:

Theorem 21.3 (Substitution rule for a definite integral) *If $g'(x)$ is a continuous function on the interval $[a, b]$ and if $f(x)$ is a continuous function that has an antiderivative on the interval $[g(a), g(b)]$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \underline{\hspace{6cm}}$$

Where does the $g'(x)$ come from? From the chain rule!

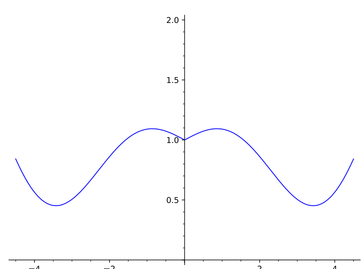
$$\begin{aligned} u = g(x) &\Rightarrow du = g'(x)\,dx \\ &\Rightarrow \frac{du}{g'(x)}\,dx \end{aligned}$$

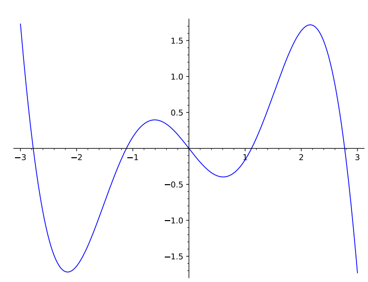
and therefore to be able to change $f(g(x))$ to $f(u)$ we must always have $g'(x)$ there.

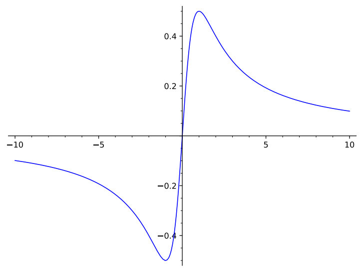
Example 21.4 Let's try this with a function that's a little bit more fun $\int \text{cotan}(ax)\,dx$ for $a \in \mathbb{R}$.

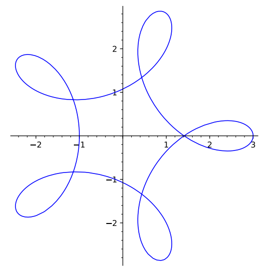
21.2 Even and odd functions

Recall that a function is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. Some examples:









Theorem 21.5 (Properties of even and odd functions) *If $f_1(x)$ and $f_2(x)$ are even functions and if $g_1(x)$ and $g_2(x)$ are odd functions, then, where the following operations are defined, we have:*

(1) $f_1(x)f_2(x)$ and $g_1(x)g_2(x)$ are ________ functions.

(2) $f_1(x)g_1(x)$ is an ______ function.

(3) $\int_{-a}^{a} f_1(x)\,dx = \underline{\qquad\qquad\qquad\qquad\qquad}$

(4) $\int_{-a}^{a} g_1(x)\,dx = \underline{\quad}$

Example 21.6 For example:

$$\int_{-a}^{a} x^2\sin(x) - x^7 + x\cos(x)\,dx = \underline{\quad}$$

21.3 Completing the square

We will look at the following three integrals:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \underline{\qquad\qquad\qquad\qquad\qquad}$$
$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \underline{\qquad\qquad\qquad\qquad\qquad\quad}$$
$$\int \frac{1}{1+x^2}\,dx = \underline{\qquad\qquad\qquad\qquad\qquad}$$

They all have the form: ______________. Therefore, we will find a way to change a polynomial ax^2+bx+c to a polynomial of the form $\pm x^2 \pm 1$. We call that: ______________________________. This consists of transforming our polynomial in the following way:

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\ &= a\left(\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right) \end{aligned}$$

But, really, it's way to difficult to memorise this formula. The only part that we really need to remember is how to choose $\frac{b^2}{4a^2}$. And for that, it's easier to start with a very easy example:

Example 21.7

$$x^2 + bx$$

From here we, we already have that $a = 1$ and $c = 0$, but, like I said, we won't look at that for the moment. We want to change $x^2 + bx$ into "a square".

Therefore, we will do the inverse. Suppose we have:

$$(x + d)^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

And in the other direction, we need to divide our b by 2 which gives us:

$$x^2 + bx =$$

Example 21.8 Let's try with some real examples:

$$x^2 + 8x = x^2 + 8x + 4^2 - 4^2 = (x + 4)^2 - 4^2$$

$$x^2 - 7x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$2x^2 + 4x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$3x^2 - 5x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

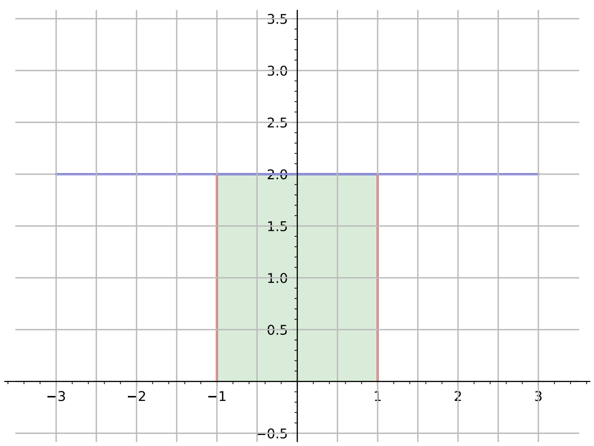
$$x^2 + 4x - 1 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Why do we do this? In order to be able to solve the integrations we had from the beginning of the section.

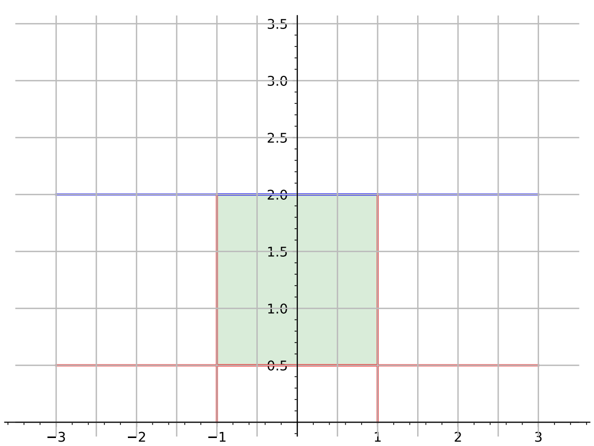
$$\int \frac{x + 2}{\sqrt{-2x^2 - 12x - 10}}\, dx$$

21.4 The area of a planar surface

Suppose that we want to find the area of a surface like a square



We want to use integration to find the area. From earlier in the course, we know taht $\int_{-1}^{1} 2\,dx$ gives us the area. But what can we do if we change the size of our square?



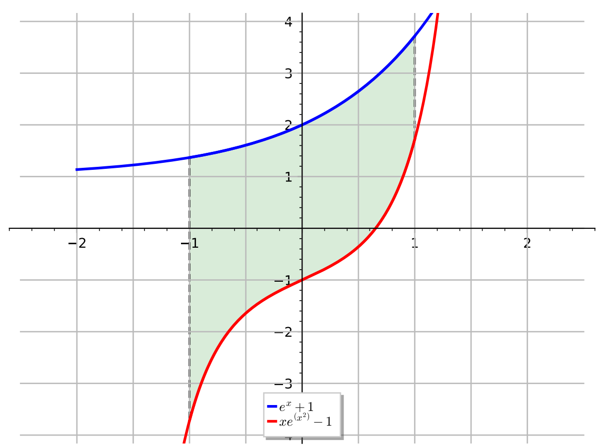
We can view it as the area of the function above minus the area of the function on the bottom. In other words

$$\int_{-1}^{1} \left(2 - \tfrac{1}{2}\right)\, dx \quad =$$

$$= 3$$

But that was way too easy. Therefore, let's try one a little bit harder.

Example 21.9 What's the area between the two curves?



The function $e^x + 1$ is in blue and the function $xe^x - 1$ is in red.

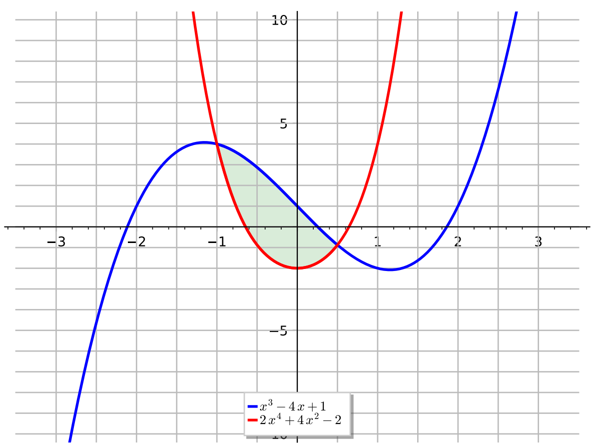
We're going to do the same thing. We will find the area under the curve on top and subtract the area under the cuve on the bottom.

Therefore, we have

$$\begin{aligned}
\int_{-1}^{1} e^x \, dx &= e^x\Big|_{-1}^{1} = e^1 - e^{-1} \\
\int_{-1}^{1} 2 \, dx &= 2x\Big|_{-1}^{1} = 2 \cdot 1 - 2 \cdot -1 = 2 + 2 = 4 \\
\int_{-1}^{1} xe^{(x^2)} \, dx &= \\
&= 0
\end{aligned}$$

Therefore, the area is $e^1 - e^{-1} + 4$.

Example 21.10 Let's do another example



Therefore, in blue we have $x^3 - 4x + 1$ and in red we have $2x^4 + 4x^2 - 2$. If we want the area in the middle, we must first find where these two curves intersect.

They intersect when ____. Therefore, we must find the roots of $2x^4 - x^3 + 4x^2 + 4x - 3$.

Then, we must use the ______________________________: p/q is a root of the polynomial $a_0 + a_1 x + \ldots + a_n x^n$ if $p|a_0$ and $q|a_n$. Therefore, we know that the roots need to be in the set: $\left\{\pm 3, \pm 1, \pm \frac{3}{2}, \pm \frac{1}{2}\right\}$.

If we look at the picture, that gives us some hints. Thus, we will look to see if -1 and $\frac{1}{2}$ are roots.

If $x = -1 \Rightarrow$ ______________________________

Therefore ____________________________.

If $x = \dfrac{1}{2} \;\Rightarrow$ ____

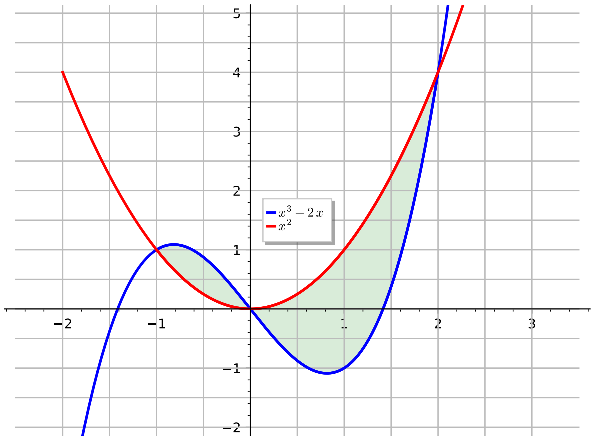
Therefore ____________________________________.

Thus, we can now find the area. But, we must know which function is on top of the other. We can see that the blue is on top, and so

$$\int_{-1}^{\frac{1}{2}} \left(x^3 - 4x + 1\right) - \left(2x^4 + 4x^2 - 2\right)\, dx \quad = \underline{\int_{-1}^{\frac{1}{2}} -2x^4 + x^3 - 4x^2 - 4x + 3\, dx \int_{-1}^{\frac{1}{2}} -2x^4}$$

$$= \tfrac{1233}{320}$$

Example 21.11 A final example. If we want the area in the following two regions:



21.5 Integration by parts

Say that $f(x) = u$ and $g(x) = v$. Then $d(uv) = f(x)g'(x) + g(x)f'(x) = udv + vdu$. Therefore, we can have $\int_a^b d(uv) = \int_a^b udv + \int_a^b vdu$. That tells us:

The indefinite version is given by:

Integration using the property above is an *integration by parts*. Let's do some examples.

Examples 21.12 (1) $\int (e^x)^2 \, dx = ?$

(2) $\int x \cos(x) \, dx =$

(3) $\int x^2 \sin(x) \, dx =$

(4) $\int e^x \sin(x)\, dx =$

Some hints: Suppose that we have $\int f(x)g(x)\, dx$. If $f(x)$ is a polynomial, we always want to try with $u = f(x)$ and $dv = g(x)\, dx$. If $f(x)$ is an exponential and $g(x)$ a trigonometric function, it doesn't matter which we choose for u and for dv.

Memory aid

L ______

I __________________________________

A ____________

T ______________________

E _____________________

Integrating trigonometric functions

Sometimes, we must use trig identities to help us solve the integral. Here's a list of identities to help you:

(1) $\sin^2(x) + \cos^2(x) =$ __

(2) $1 + \tan^2(x) =$ ____________

(3) $1 + \cot^2(x) =$ ____________

(4) $\sin(x)\cos(y) = \frac{1}{2}\left(\sin(x-y) + \sin(x+y)\right)$

(5) $\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

(6) $\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$

(7) $\sin(x \pm y) =$ ______________________________

(8) $\cos(x \pm y) =$ ______________________________

We can use this list to have simpler identities:

(1) $\sin^2(x) =$ __________

(2) $\cos^2(x) =$ __________

(3) $\sin(2x) =$ _______________

How does that help us?

Example 21.13 Try to solve the integral: $\int \sin(32x)\cos(3501x)\,dx$. We use integration by parts!

$$u = \sin(32x) \qquad dv = \cos(3501x)\,dx$$
$$du = \cos(32x) \qquad v = \frac{\sin(3501x)}{3501}$$

$$\int \sin(32x)\cos(3501x)\,dx = \underline{\hspace{8cm}}$$

We have the same problem! Nothing has changed. ☹ Therefore, we will use a trig identity.

$$\begin{aligned}
\sin(32x)\cos(3501x) &= \underline{\hspace{8cm}} \\
&= \underline{\hspace{7cm}}
\end{aligned}$$

$$\begin{aligned}
\int \sin(32x)\cos(3501x)\,dx &= \underline{\hspace{8cm}} \\
&= \frac{-\cos(-3469x)}{2\cdot -3469} + \frac{-\cos(3533x)}{2\cdot 3533} + c \\
&= \frac{\cos(-3469x)}{6938} + \frac{-\cos(3533x)}{7066} + c
\end{aligned}$$

Integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$

Suppose we have an integral of the form: $\int \sin^m(x)\cos^n(x)\,dx$. There are different approaches depending on the values of m and n.

(1) If m and n are odd, we have a way to find it. Suppose $m = 2k+1$.

We can use integration by substitution!

$$u = \cos(x) \qquad du = -\sin(x)\, dx$$

$$\begin{aligned}
\int \sin^m(x)\cos^n(x)\,dx &= \int \sin^{2k+1}(x) u^n \frac{du}{-\sin(x)} \\
&= -\int \sin^{2k}(x) u^n du \\
&= -\int (\sin^2(x))^k u^n du \\
&= -\int (1-\cos^2(x))^k u^n du \\
&= -\int (1-u^2)^k u^n du
\end{aligned}$$

Therefore, trying with a real example: $\int \cos^5(x)\sin^2(x)\,dx$. We have $5 = 2\cdot 2+1$, which is odd!

$$u = \sin(x) \qquad du = \cos(x)\, dx$$

$$\begin{aligned}
\int \cos^5(x)\sin^2(x)\,dx &= \underline{\hspace{10cm}} \\
&= \int \left(\cos^2(x)\right)^2 u^2 du \\
&= \underline{\hspace{9cm}} \\
&= \int \left(1-u^2\right)^2 u^2 du \\
&= \underline{\hspace{9cm}} \\
&= \int u^2 - 2u^4 + u^6 du \\
&= \int u^2 du - 2\int u^4 du + \int u^6 du \\
&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + c \\
&= \frac{\sin^3(x)}{3} - \frac{2\sin^5(x)}{5} + \frac{\sin^7(x)}{7} + c
\end{aligned}$$

And something harder: $\int \sin^3(x)\cos^{(-3/7)}(x)\,dx$

(2) If m and n are even and non-negative. We will use the three properties we saw earlier:

(a) $\sin^2(x) = \frac{1-\cos(2x)}{2}$

(b) $\cos^2(x) = \frac{1+\cos(2x)}{2}$

(c) $\sin(2x) = 2\sin(x)\cos(x)$

Using these formulae, we can perform manipulations in order to decrease the degree.

Example 21.14 Let's see an example: $\int \sin^4(x)\cos^2(x)\,dx$

$$\int \sin^4(x)\cos^2(x)\,dx = \underline{\qquad\qquad\qquad\qquad}$$

$$= \frac{x}{16} - \frac{\sin(4x)}{64} + c - \frac{1}{8}\int \sin^2(2x)\cos(2x)\,dx$$

Therefore, we must solve $\int \sin^2(2x)\cos(2x)\,dx$

$$u = \sin(2x) \qquad du = 2\cos(2x)\,dx$$

Therefore, we have

21.6 Secants, cosecants, tangents, cotangents!

Let's start with $\int \tan^n(ax)\,dx$ where $n \geq 1$. In general, we have

$$\begin{aligned}\int \tan^n(ax)\,dx &= \int \tan^2(ax)\tan^{n-2}(ax)\,dx \\ &= \underline{\hspace{8cm}} \\ &= \int \sec^2(ax)\tan^{n-2}(ax)\,dx - \int \tan^{n-2}(ax)\,dx\end{aligned}$$

This integral doesn't seem easier than the previous one. Therefore, we try to use one of the methods that we have already seen in this course: ____. Let's start with ____________________. Set $u =$ _____________, then $du =$ ____

____________________. From which

$$\begin{aligned}\int \sec^2(ax)\tan^{n-2}(ax)\,dx - \int \tan^{n-2}(ax)\,dx &= \underline{\hspace{6cm}}\\ &= \underline{\hspace{6cm}}\\ &= \underline{\hspace{6cm}}\\ &= \frac{\tan^{n-1}(ax)}{a(n-1)} + c - \int \tan^{n-2}(ax)\,dx\end{aligned}$$

Therefore

$$\int \tan^n(ax)\,dx = \underline{\hspace{10cm}}$$

which is much better!

Example 21.15 Let's see that with an example: $\int \tan^5(2x)\,dx$.

We can use a similar technique for the secant: $\int \sec^n(ax)\,dx$.

Example 21.16 Let's solve $\int \sec^8(x)\,dx$

This method does not work when n is even. If n is odd, we must use the formula of integration by parts.

21.7 Integration by trigonometric substitution

The idea now is to find a way to integrate integrals which look like: ________________________. For the moment, no method we have see will give us an easy way to solve it. But, we can use trig identities to solve them.

Example 21.17 If we have $\int \sqrt{a^2 - x^2}\,dx$ we will use the fact that $\sin(x)^2 + \cos(x)^2 = 1$ to solve the integral. How does that help us? Because we can play with the formula!

In particular, $\cos(x)^2 = 1 - \sin(x)^2$, therefore if we replace x by ____________ we will have something simpler.

$\sqrt{a^2 - x^2} =$ ____

Taking the derivative, we have $dx =$ ________________! Therefore, $\int \sqrt{a^2 - x^2}\,dx = \int a^2 \cos^2(\theta)\,d\theta$.

And from here we already saw how to do that:

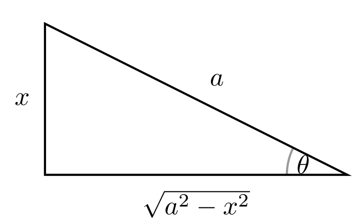
$$\int \sqrt{a^2 - x^2}\,dx = \int a^2 \cos^2(\theta)\,d\theta$$

$$= \underline{\hspace{6cm}}$$

$$= \underline{\hspace{6cm}}.$$

BUT! We have another problem now. We want the formula with x. Therefore, we must find the value of θ and $\sin(2\theta)$ in terms of x.

If $x = a\sin(\theta)$ then $\arcsin(\frac{x}{a}) = \theta$. For $\sin 2\theta$ we have $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Using the triangle:



we have $\sin(\theta) = \underline{\quad}$ and $\cos(\theta) = \underline{\qquad\qquad}$. Therefore,

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx &= \frac{a^2}{2}\theta + \frac{1}{2}\sin(2\theta) + c \\ &= \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\ &= \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.\end{aligned}$$

But we have many identities that we can use to help us!

$$\sin(x)^2 + \cos(x)^2 = 1 \qquad 1 + \tan(x)^2 = \sec(x)^2$$

For each case we use a different substitution:

$a^2 - x^2$	$x = \underline{\qquad\qquad}$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$a^2 + x^2$	$x = \underline{\qquad\qquad}$	$\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$
$x^2 - a^2$	$x = \underline{\qquad\qquad}$	$\begin{cases}\theta \in \left[0, \frac{\pi}{2}\right[& \text{when } x \geq a \\ \theta \in \left]\frac{\pi}{2}, \pi\right] & \text{when } x \leq -a\end{cases}$

Let's do some examples.

Examples 21.18 (1) $\int \left(25 + (x-3)^2\right)^{-3/2} dx$

The formula $25 + (x-3)^2$ is like $1 + x^2$, therefore we're going to use $x = \underline{\qquad\qquad}$ as substitution.

(2) $\int \frac{1}{x\sqrt{2x^2 - 16}}\, dx$

Say that $\sqrt{2}x > 4$ and that $\underline{\qquad\qquad\qquad}$

21.8 Integrating expressions which contain quadratic functions

We already have all the tools we need to integrate functions which contain $ax^2 + bx + c$.

Remember how to complete the square.

$$
\begin{aligned}
ax^2 + bx + c &= \underline{\hspace{6cm}} \\
&= \underline{\hspace{10cm}} \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right)
\end{aligned}
$$

From here, we can use the previous methods to find the integral.

Example 21.19 Let's see an example: $\int \frac{1}{\sqrt{x^2-6x+14}}\,dx$.

$$
\begin{aligned}
x^2 - 6x + 14 &= \underline{\hspace{10cm}} \\
\int \frac{1}{\sqrt{x^2-6x+14}}\,dx &= \\
&\quad \underline{\hspace{6cm}} \\
&\Rightarrow \underline{\hspace{11cm}} \\
\therefore \quad &= \int \frac{\sqrt{5}\sec^2(\theta)}{\sqrt{5\tan^2(\theta)+5}}\,d\theta \\
&= \int \frac{\sqrt{5}\sec^2(\theta)}{\sqrt{5}\sqrt{\tan^2(\theta)+1}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \sec(\theta)\,d\theta \\
&= \underline{\hspace{7cm}} \\
&= \ln\left(\left|\frac{\sqrt{(x-3)^2+5}}{\sqrt{5}} + \frac{x-3}{\sqrt{5}}\right|\right) + c
\end{aligned}
$$

21.9 Integrating rational functions by partial fraction decomposition

If our integral contains a fraction whose numerator and denominator are polynomials, it's sometimes useful to use partial fraction decomposition. Let's see this in an example.

Example 21.20 Given $\int \frac{3x^2+x+2}{x^3-x^2-x+1}\,dx$

Chapter 6

Sequences and series of functions

22 Sequences of functions

In section 5 we studied sequences of numbers, but what happens if we take sequences of functions? Some things get more complicated, but a lot of things tend to be almost the same! This is a good time to go back and review chapter 2 as we'll be doing a lot of similar things.

22.1 Definitions

Definition 22.1 A *sequence of functions* is a sequence $\{f_n\}$ on A such that each f_n is a function $f_n : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$.

Example 22.2 The following are sequences of functions:

(1) $\{f_n\}$ where $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$,

(2) $\{g_n\}$ where $g_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$.

(3) Make your own: ____________________________

22.2 Bounds

Definition 22.3 We say that a sequence of functions $\{f_n\}$ on $A \subseteq \mathbb{R}$ is *pointwise bounded* if the sequence $\{f_n(x)\}$ is bounded for every $x \in A$. In other words, for each $x \in A$, there exists a number $M(x)$, such that $|f_n(x)| \leq M(x)$.

A sequence of functions $\{f_n\}$ on $A \subseteq \mathbb{R}$ is said to be *uniformly bounded* if there exists M such that

$$|f_n(x)| \leq M \quad \forall x \in A \text{ and } n \in \mathbb{N}$$

Definition 22.4 We say that a sequence of functions $\{f_n\}$ on $A \subseteq \mathbb{R}$ is *monotonically increasing* (resp. decreasing) if $f_n(x) \leq f_{n+1}(x)$ (resp. $f_n(x) \geq f_{n+1}(x)$) for all $x \in A$ and all $n \in \mathbb{N}$.

22.3 Convergence and limits

As we did before, we'll want to know if these sequences converge and, if they converge, what their limits are. Since we're working with functions, there's actually two different notions of convergence which seem similar, but are *not* the same.

Definition 22.5 Given $x \in A \subseteq \mathbb{R}$, we say that the sequence of functions $\{f_n\}$ on $A \subseteq \mathbb{R}$ is *pointwise convergent* at x if the sequence $\{f_n(x)\}$ is convergent. We let $f(x)$ denote the *pointwise limit* of the sequence $\{f_n(x)\}$. In other words:

Wikipedia: pointwise convergent

$$f(x) = \underline{\hspace{6cm}} \qquad x \in A.$$

Definition 22.6 The set C of all points $x \in A \subseteq \mathbb{R}$ such that $\{f_n\}$ converges pointwise at x is called the *set of points of pointwise convergence*.

$$C = \underline{\hspace{16cm}}$$

Definition 22.7 Given a sequence of functions $\{f_n\}$ on $A \subseteq \mathbb{R}$, we say that the sequence *converges uniformly* in $X \subseteq A$ if there exists a function $f : X \to \mathbb{R}$ such that

Wikipedia: converges uniformly

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \text{ and } \forall x \in X, |f_n(x) - f(x)| < \epsilon.$$

Proposition 22.8 *Given a sequence of functions $\{f_n\}$ and C its set of points of pointwise convergence. For each $n \in \mathbb{N}$ we let*

$$m_n = \sup\{|f_n(x) - f(x)| \mid x \in C\}$$

The following are equivalent:

(1) The sequence $\{f_n\}$ converges uniformly on C towards f.

(2) There exists an $N \in \mathbb{N}$ such that $m_n \in \mathbb{R}$ for all $n \geq N$ and the sequence $\{m_n\}$ converges to 0.

Theorem 22.9 (Uniform limit theorem) *Given $X \subseteq A \subset \mathbb{R}$ and a sequence of functions $\{f_n\}$ on A and that converge uniformly at X to the function f. If every f_n is continuous then f must also be continuous.*

Example 22.10 State if the following sequence of functions converges pointwise and/or uniformly.

$$f_n(x) = x^n,\ x \in [0, 1]$$

Example 22.11 State if the following sequence of functions converges pointwise and/or uniformly.

$$f_n(x) = x^n,\ x \in \left[0, \frac{1}{2}\right]$$

23 Series of functions

Given a sequence of functions $\{f_n\}$ on $A \subseteq \mathbb{R}$, we can consider the seqeunce of partial sums $\{S_n\}$ defined by

$$S_n = \sum_{k=1}^{n} f_k \qquad \forall n \in \mathbb{N}.$$

in other words,

$$S_n(x) = \sum_{k=1}^{n} f_k(x) \qquad \forall x \in A, \forall n \in \mathbb{N}$$

Definition 23.1 Then the series $S(x) = \sum_{k=1}^{\infty} f_k(x)$ *converges (pointwise/uniformly)* to a function $S : A \to \mathbb{R}$ if $\{S_n\}$ converges (pointwise/uniformly) to $\{S\}$. We call $S = \sum f_n$ a *series of functions*.

The sequence $\{f_n\}$ is the *general term* of the series $\sum f_n$ and the sum S_n is the n-th partial sum of the series $\sum f_n$ with sequence of partial sums $\{S_n\}$.

23.1 Convergence

The series of functions $\sum f_n$ converges to a point $x \in A$ when the sequence of partial sums $\{S_n(x)\}$ converges. This is equivalent to stating that the series $\sum f_n(x)$ is convergent.

Definition 23.2 Given nonempty $C \subseteq A$ and $f : C \to \mathbb{R}$. The series $\sum f_n$ *converges pointwise* on C when the series $\sum f_n(x)$ converges with limit $f(x)$.

$$f(x) = \lim_{n\to\infty} S_n(x)$$

with the sum function of the series given by:

$$f = \sum_{n=1}^{\infty} f_n(x)$$

Example 23.3 For $\alpha > 0$, define $u_n : [0, \infty) \to \mathbb{R}$ for all $n \in \mathbb{N}$ such that:

$$u_n(x) = \frac{x^\alpha}{1 + n^2x^2}$$

Check for what values, if any, the series of functions $\sum u_n$ converges pointwise on $[0, \infty)$ for all values of α.

From the general theory on convergent series, we know that if a series of functions is pointwise convergent, then its general term has to converge pointwise to 0.

Definition 23.4 The series $\sum f_n$ is *uniformly convergent* on $C \subseteq A$ when

$$\forall \epsilon > 0,\ \exists m \in \mathbb{N} \text{ such that } n \geq m \text{ implies } \left| f(x) - \sum_{k=1}^{n} f_k(x) \right| < \epsilon \quad \forall x \in C$$

In other words, the series is uniformly convergent when the sequence of partial sums is uniformly convergent.

Corollary 23.5 *A necessary condition that the series of functions $\sum f_n$ is uniformly convergent in a set A is that the sequence of functions $\{f_n\}$ is uniformly convergent to 0 en A.*

Definition 23.6 Given a series of functions $\sum f_n$ on a set X. If the series $\sum |f_n|$ is pointwise convergent on X we say that the original series is *absolutely (pointwise) convergent* on X.

Corollary 23.7 *Every absolutely convergent series is pointwise convergent. Toda serie absolutamente convergente es puntualmente convergente.*

Definition 23.8 The *set of points of absolute convergence* of a series of functions $\sum f_n$ is the set

$$D = \left\{ x \in A \mid \sum |f_n(x)| \text{ is convergent} \right\}$$

Wikipedia: normally convergent

Definition 23.9 Given a series of functions $\sum f_n$ on a set X. We say that the series is *normally convergent* on X if there exists a convergent series of non-negative numbers $\sum m_n$ such that for all $n \in \mathbb{N}$ we have:

$$|f_n(x)| \leq m_n \qquad \forall x \in X.$$

Wikipedia: Weierstrass M-test

Theorem 23.10 (Weierstrass M-test) *Given a series of functions $\sum f_n$ and A a set such that for all $x \in A$ and $n \in \mathbb{N}$ we have that $|f_n(x)| \leq M_n$, where the series $\sum M_n$ converges. Then $\sum f_n$ converges uniformly and absolutely on A.*

Example 23.11 Prove that the following series of functions converges uniformly on $\mathbb{R}$:

$$\sum_{n=1}^{\infty} \frac{\sin^n(x)}{n^{\frac{5}{2}}}$$

Example 23.12 Does the following sum converge? If so, to what value does it converge?

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

23.2 Power series

Definition 23.13 A series of functions of the following form is known as a *power series* centered at $x = c$.

Wikipedia: power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

where a_n is a function of n and $c \in \mathbb{R}$.

Example 23.14 Here are some examples:

(1) $\sum_{i=0}^{\infty} x^i$

(2) $\sum_{i=0}^{\infty} \frac{(x-1)^i}{i}$

(3) $\sum_{i=0}^{\infty} \frac{2i}{\sqrt{i+1}} (x-2)^i$

(4) Make your own: ____________________________

If we look at the first example, we can see that if $|x| < 1$, then the series will _______________ and if $|x| > 1$, then the series will ____________.

$$\sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots$$
$$\sum_{i=0}^{\infty} (2)^i = 1 + 2 + 2^2 + 2^3 + \dots$$

Therefore, we can see that, according to our choice of x, the series can either diverge or converge. The set I of *all* points x such that the power series converges is called the interval of convergence. Every power series has at least one point which converges since if $x = c$ then $(x - c) = 0$ and our power series is equal to ____.

Notice how when $x = 1$, we don't know whether it converges or not. This point is called the radius of convergence. We define it in the following way:

Definition 23.15 The *radius of convergence* is the number $r \in \mathbb{R}$ where

(1) If the series converges *only* for $x = c$ then $r = 0$.

(2) If the series converges for all x such that $|x - c| < r$ and diverges for $|x - c| > r$.

(3) If the series converges for *all* x, then $r = \infty$.

The *interval of convergence* is then the interval of all points where the power series converges.

Theorem 23.16 *Given a power series* $\sum a_i(x-c)^i$*, then the radius of convergence is given by* $r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ *when this limit exists or is equal to* ∞*.*

Example 23.17 Find the radius of convergence and interval of convergence of

$$\sum_{n=1}^{\infty}(2^n + 4^n)x^n$$

Definition 23.18 Given I, an open interval in $\mathbb{R}$, and f a real function defined on I. We say thhat f is *analytic* at a point $x_0 \in I$ if there exists a real number $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset I$ and the power series

Wikipedia: analytic

$\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for all points in $(x_0-\delta, x_0+\delta)$, such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \qquad x \in (x_0-\delta, x_0+\delta)$$

In other words, that the power series converges at x_0 and coincides with the function f at said point.

23.3 Taylor and Maclaurin series

We can convert functions to power series in order to more easily integrate them. We do this using something called the Taylor series. But first we need a theorem.

Theorem 23.19 *If the power series $\sum a_i(x-c)^i$ has radius of convergence $r > 0$, then the function defined by $f(x) = \sum a_i(x-c)^i$ is differentiable and integrable on the open interval $(c-r, c+r)$, i.e., f is analytic at c. Furthermore*

$$f'(x) = \underline{\hspace{6cm}}$$

and

$$\int f(x)\,dx = \underline{\hspace{7cm}}.$$

The radius of convergence of these two new power series is also equal to r.

Now let's do something a little weird. What happens if we keep taking derivatives?

$$f(x) = \sum_{i=0}^{\infty} a_i(x-c)^i \Rightarrow f(c) = \underline{\hspace{1cm}}$$

$$f'(x) = \sum_{i=1}^{\infty} i a_i(x-c)^{i-1} \Rightarrow f'(c) = \underline{\hspace{1cm}}$$

$$f''(x) = \sum_{i=2}^{\infty} i \cdot i-1 \cdot a_i(x-c)^{i-2} \Rightarrow f''(c) = \underline{\hspace{1.5cm}}$$

$$f'''(x) = \sum_{i=3}^{\infty} i \cdot i-1 \cdot i-2 \cdot a_i(x-c)^{i-3} \Rightarrow f'''(c) = \underline{\hspace{3cm}}$$

$$f^{(4)}(x) = \sum_{i=4}^{\infty} i \cdot i-1 \cdot i-2 \cdot i-3 \cdot a_i(x-c)^{i-4} \Rightarrow f''(c) = \underline{\hspace{3.5cm}}$$

We can then replace each a_i in the summation for $f(x)$ with the appropriate derivative:

$$f(x) = \underline{\hspace{6cm}}$$

Wikipedia: Taylor series

We call this series the *Taylor series* centered at c. If $c = 0$, then this series is sometimes also known as the *Maclaurin series*.

Example 23.20 Let's look at an example for $f(x) = e^x$.

What's the radius of convergence?

Here are more Taylor series centered at $c = 0$.

(1) $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad x \in \mathbb{R}$,

(2) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad x \in \mathbb{R}$,

(3) $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \quad x \in (-1,1)$

Appendix A

Numerical sets

1 Sets

1.1 Definition

Definition 1.1 A *set* is a *well-defined* collection of elements. By "well-defined" we mean that there is a well established rule for whether something is in the set or not.

Example 1.2

- The set of all tall people is *not* a set. .
- The set of all people who are taller than 1.5m *is* is a set.

The elements which are part of a set can be absolutely anything. They can be numbers, sets, names, movies, memories, etc.

1.2 Notation

A set is written using curly brackets: $\{\}$. We separate each element in the set by a comma. We will normally use capital letters to represent sets and lower case letters to represent elements of the set.

Example 1.3

- $A = \{1, 4, 7, 9\}$
- $B = \{x, y, z\}$
- $C = \{\text{Victor, Leticia, Juan, Cristina}\}$

To define a set more formally, we need to formalize things.

Definition 1.4 If an element a is *contained in* or *is an element of* a set A, we denote this as:

$$a \in A$$

If this is not the case (a is not contained in A), then we write:

$$a \notin A$$

We can define a set using a sequence of elements or by giving a rule.

Example 1.5

$$A = \{x \mid x \text{ is a vowel}\}$$
$$A = \{a, e, i, o, u\}$$

$$B = \{x \mid x \in \mathbb{Z},\, 1 \leq x \leq 4\}$$
$$B = \{1, 2, 3, 4\}$$

where " | " is read as "such that".

2 Important sets

The most important sets are the following:

Empty set $\varnothing$ (also written as $\{\}$)

Natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$

Integers $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

Rational numbers $\mathbb{Q} = \left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \neq 0\right\}$

Real numbers $\mathbb{R}$

Complex numbers $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ $(i^2 = -1)$

Universal set $\mathcal{U}$

Each of the above sets have important properties which are exhaustively used in mathematics in order to prove certain arguments and for reasoning.

3 Graphical representation

Wikipedia: Venn diagram

Definition 3.1 A *Venn diagram* is a graphic used to visualize sets and the interactions between them.

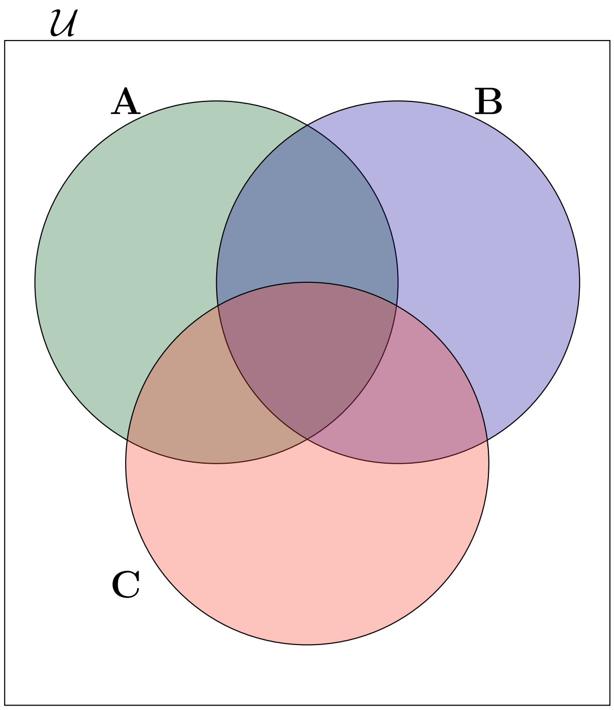


Figure A.1: Venn diagram

Example 3.2 If we consider the sets $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, its Venn diagram is:

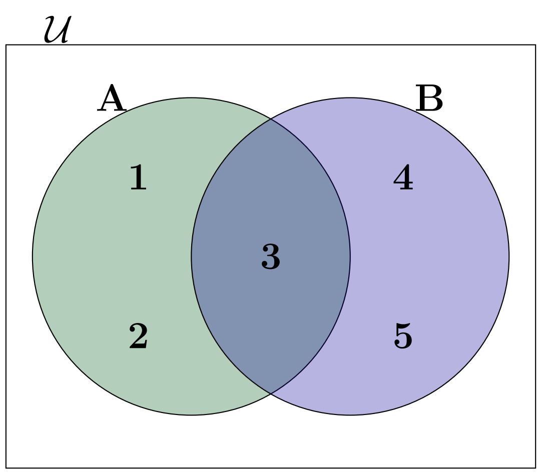


Figure A.2: Example of Venn diagram

where the shared element 3 appears in the intersection of the two sets.

4 Subsets

Definition 4.1 Let A and B be two sets. We say that A is a *subset* of B, written as $A \subseteq B$, if every element of A also appears in B. In other words $x \in A$ implies $x \in B$. The Venn diagram of this can be seen as:

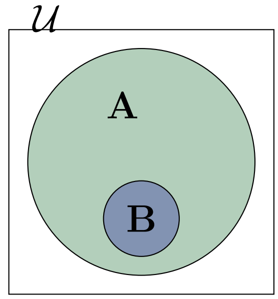


Figure A.3: Example of subsets

The important sets from previously can be viewed as subsets of each other.

$$\varnothing \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

5 Operations

The interaction between various sets are known as *operations*.

5.1 Intersection

Wikipedia: intersection

Definition 5.1 The *intersection* of two sets A and B is the set of all elements which appear in both sets. We denote this set as:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

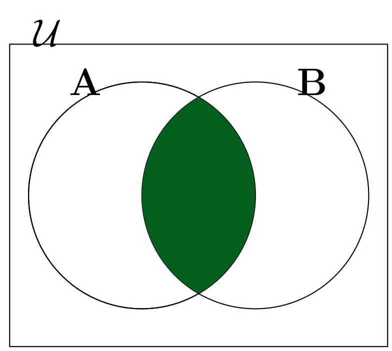


Figure A.4: Intersection: $A \cap B$

5.2 Union

Definition 5.2 The *union* of two sets A and B is the set of all elements which appear in at least one of the two sets. We denote this set as:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

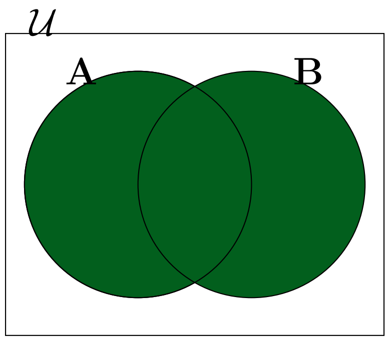


Figure A.5: Union: $A \cup B$

5.3 Difference

Definition 5.3 The *difference* of two sets A and B is the set of all elements which appear in A and don't appear in B. We denote this set as:

$$A - B = \{x \mid x \in A, x \notin B\}$$

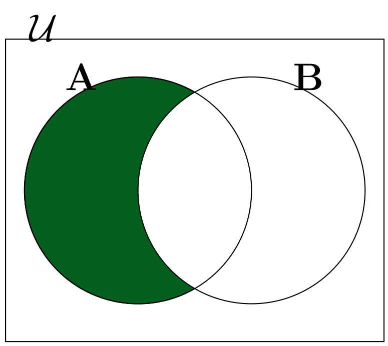


Figure A.6: Difference: $A - B$

5.4 Symmetric difference

Definition 5.4 The *symmetric difference* of two sets A and B is the set of all elements which appear in one of the sets, but not both. We denote this set as:

Wikipedia: symmetric difference

$$A \triangle B = \{x \mid (x \in A, x \notin B) \text{ or } (x \in B, x \notin A)\} = (A - B) \cup (B - A)$$

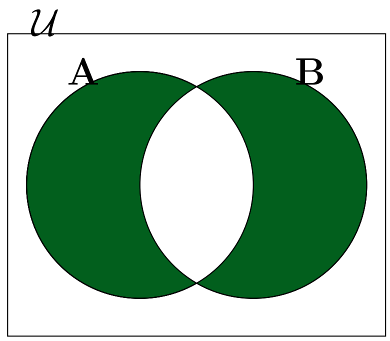


Figure A.7: Symmetric difference: $A \triangle B$

5.5 Complement

Definition 5.5 Let A be a subset of another set B, $A \subseteq B$. The *complement* of A is the set of all elements in B, but not A. We denote this set as:

$$\overline{A} = \{x \mid x \notin A\} = B - A$$

The complement is only defined if A is a subset of some other set B.

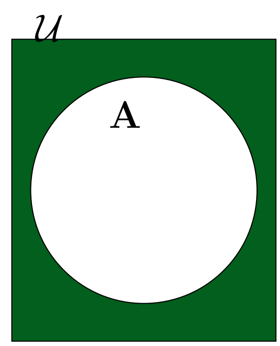


Figure A.8: Complement: $\overline{A}$

6 Properties

Given two sets A, B and a universal set $\mathcal{U}$. **Idempotent property:**

$$A \cap A = A \quad A \cup A = A$$

Commutative property:

$$A \cap B = B \cap A \quad A \cup B = B \cup A \quad A \triangle B = B \triangle A$$

Associative property:

$$
\begin{aligned}
(A \cap B) \cap C &= A \cap (B \cap C), \\
(A \cup B) \cup C &= A \cup (B \cup C), \\
(A \triangle B) \triangle C &= A \triangle (B \triangle C).
\end{aligned}
$$

Absorption property:

$$A \cap (A \cup B) = A \quad A \cup (A \cap B) = A$$

Distributive property:

$$
\begin{aligned}
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)
\end{aligned}
$$

Involution law:

$$\overline{\overline{A}} = A$$

De Morgan's laws:

$$
\begin{aligned}
\overline{A \cup B} &= \overline{A} \cap \overline{B} \\
\overline{A \cap B} &= \overline{A} \cup \overline{B}
\end{aligned}
$$

Complement laws:

$$
\begin{gathered}
A \cup \overline{A} = \mathcal{U} \quad A \cap \overline{A} = \varnothing \\
\overline{\varnothing} = \mathcal{U} \quad \overline{\mathcal{U}} = \varnothing
\end{gathered}
$$

Additional laws:

$$
\begin{array}{ccc}
A \cup \varnothing = A & A \cup A = A & A \cup \mathcal{U} = \mathcal{U} \\
A \cap \varnothing = \varnothing & A \cap A = \varnothing & A \cap \mathcal{U} = A \\
A \triangle \varnothing = A & A \triangle A \varnothing & A \triangle \mathcal{U} = \overline{A}
\end{array}
$$

7 Cardinality

Definition 7.1 The *cardinal* of a set A, written $|A|$, is the number of elements in A.

Example 7.2 Consider the set $A = \{a, b, c, d\}$. Then $|A| = 4$.

8 Power sets

Wikipedia: power set

Definition 8.1 The *power set* of a set A, written $\mathcal{P}(A)$, is the set of all subsets of A.

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}$$

Example 8.2 Given the set $A = \{1, 2, 3\}$.

Then the power set of A is:

$$\mathcal{P}(A) =$$

Noting that every either element is either in a subset or not, this means that:

$$|\mathcal{P}(A)| = 2^{|A|}.$$

9 Cartesian product

9.1 Definition

Definition 9.1 Given two sets A and B. The set formed by taking pairs of *ordered* elements whose first component is in A and whose second component is in B is called the *cartesian product* of A and B and is denoted:

$$A \times B = \{(x, y) \mid x \in A, \, y \in B\}$$

Example 9.2 Find the cartesian product of $A = \{1, 2, 3\}$ and $B = \{a, b\}$.

$$A \times B =$$

9.2 Properties

Empty set:

$$A \times \varnothing = \varnothing \times A = \varnothing$$

Not commutative:

$$A \times B \neq B \times A$$

Not associative:

$$A \times (B \times C) \neq (A \times B) \times C$$

Cardinality:

$$|A_1 \times \ldots \times A_n| = |A_1| \cdot \ldots \cdot |A_n|$$

Appendix B

Partial fraction decomposition

If we have a fraction of two polynomials $\frac{P(x)}{Q(x)}$, then we sometimes want to split the fraction into a sum of simpler fractions where the denominator is linear or quadratic. We do this in the following way.

10 Factor

The first step that we do is to factor $Q(x)$ into linear and quadratic polynomials. In other words

$Q(x) =$ ____

This is best seen with an example.

Example 10.1 Given

$$Q(x) = 2x^5 + 5x^4 - 12x^3 - 14x^2 + 34x - 15.$$

We must use the ________________________________.

We have $-15 = a_0$ and $2 = a_n$. Therefore p/q may be: ________________________________

_____________________________________. Let's try with the simplest: 1.

$Q(1) =$ ____

Therefore, $(x-1)|Q(x)$ and

$Q(x) =$ ____

Let's try 1 again:

$Q(1) =$ ____.

Therefore:

$Q(x) =$ ____

Then, we try 1 one more time.

$Q(1) =$ ____________________________________.

Thus:

$Q(x) =$ ____

But, from here, it's evident that 1 (nor any positive number) will work because we will always get something strictly positive! Therefore, let's try -1.

$Q(-1) =$ ___________________________________.

Nope! Let's try, -3.

$Q(-3) =$ ____

Therefore:

$$Q(x) = (x-1)^3(x+3)\,(2x+5)$$

11 Decompose

The second step is to expand our quotient with respect to the denominator.

$$\frac{P(x)}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$$

$$\frac{P(x)}{(ax^2+bx+c)^n} = \frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

If we have a few, we add them together.

$$\frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b}+\frac{B}{cx+d}$$

Examples 11.1 (1) In our example: $\frac{P(x)}{2x^5+5x^4-12x^3-14x^2+34x-15}$.

$$\frac{P(x)}{(x-1)^3(x+3)(2x+5)} = \frac{A}{x+3}+\frac{B}{2x+5}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$$

(2) $\frac{P(x)}{(x^2+1)^2(2x+3)}$

(3) $\frac{P(x)}{x^2(x+1)^3\left(x-\frac{5}{2}\right)(x^2+1)^3(x^2+2x+8)}$.

The third step is to solve for the upper-case variables. But, how can we find the upper-case variables? If we don't have powers, it's easy:

Example 11.2 $\frac{12x+3}{x(x-1)} =$ ________________.

$=$ ____________________________________

$12x+3 =$ ____________________________

Si $x = 0$then ______________

Si $x = 1$then _____________

Alors $A =$ ______ et $B =$ ____. Donc

$$\frac{12x+3}{x(x-1)} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Example 11.3 $\frac{2x^2-15x+3}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$.

$$= \underline{\hspace{8cm}}$$

$$\begin{aligned} 2x^2 - 15x + 3 &= A(x+3) + B(x-1) \\ \text{Si } x = 1\text{then } &\underline{\hspace{8cm}} \\ \text{Si } x = -3\text{then } &\underline{\hspace{8cm}} \end{aligned}$$

Therefore, $A =$ _____ et $B =$ ______ giving us

$$\frac{2x^2 - 15x + 3}{(x-1)(x+3)} = \underline{\hspace{5cm}}$$

Let's try some more complicated examples.

Example 11.4

$$\frac{x^2 + x}{(x+1)^3} = \underline{\hspace{7cm}}$$

Starting in a similar way:

$$\begin{aligned} x^2 + x &= \underline{\hspace{6cm}} \\ C &= \underline{\hspace{0.5cm}} \end{aligned}$$

But, we don't know what to substitute for the x in order to find A and B! So, we're going to expand and compare coefficients from one side to the other.

$$\begin{aligned} x^2 + x &= Ax^2 + 2Ax + A + Bx + B - 1 \\ &= \underline{\hspace{8cm}} \\ A &= \underline{\hspace{0.5cm}} \\ B &= \underline{\hspace{0.5cm}} \end{aligned}$$

Therefore,

Example 11.5 How about an example in which the denominator has more than one term and has powers.

$$\frac{x^2+x}{x^2(x+1)^3} = \underline{\hspace{10cm}}$$

Plugging in 0 and -1 then gives us:

$$\begin{aligned} x^2+x &= \underline{\hspace{15cm}} \\ B &= \underline{\hspace{0.5cm}} \\ E &= \underline{\hspace{0.5cm}} \end{aligned}$$

For A, C and D, we do the same as before.

$$\begin{aligned} x^2+x &= Ax(x+1)^3 + Cx^2(x+1)^2 + Dx^2(x+1) \\ &= Ax^4 + 3Ax^3 + 3Ax^2 + Ax + Cx^4 + 2Cx^3 + Cx^2 + Dx^3 + Dx^2 \\ &= \underline{\hspace{15cm}} \\ A &= \underline{\hspace{0.5cm}} \\ C &= \underline{\hspace{0.5cm}} \\ D &= \underline{\hspace{0.5cm}} \end{aligned}$$

Therefore,

Example 11.6 Let's start with the example from the very beginning:

$$\frac{2x^4 + x^3 + 17x - 20}{(x-1)^3(x+3)(2x+5)} = \underline{\hspace{8cm}}$$

Index

Bibliography

The following books were used to create this text.

[1] PhD. Thomas Ian Ashley. *Cálculo 1*. Universidad de Loyola, 2025.

[2] James Stewart. *Calculus: Early Transcendentals*. Brooks Cole, 2015.