Bruhat Preclosure

(arXiv: 2512.08711)

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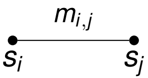
Coxeter groups

- *Coxeter System* (W, S)

$$W := \langle S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

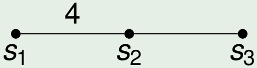
  $m_{i,j} \in \mathbb{N}^{\star} \cup \{\infty\}$ and $m_{i,j} = 1$ only if $i = j$.
- *Coxeter diagram* Γ_W: vertices S and edges labelled $m_{i,j}$ when $m_{i,j} > 3$.



Example

$$W_{B_3} = \left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \right\rangle$$

Γ_{B_3} :



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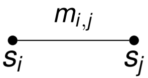
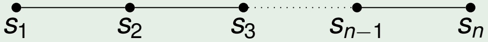


Example

$W_{A_n} = S_{n+1}$, symmetric group.

Γ_{A_n} :



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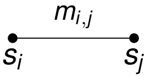
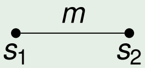


Example

$W_{I_2(m)} = \mathcal{D}(m)$, dihedral group of order $2m$.

$\Gamma_{I_2(m)}$:



Coxeter groups

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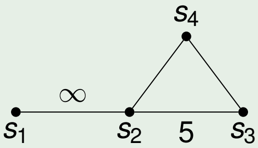
Example

W, an infinite Coxeter group.

Γ :



Words

(W, S) a Coxeter system.

- $w = s_1 \dots s_n \in W$ for $s_i \in S$.
- w has *length* n, $\ell(w) = n$, if n is minimal.
- If $\ell(w) = n$ then $w = s_1 \dots s_n$ is a *reduced expression*.

Example

Γ_{A_2} : 

Then $\ell(stst) = 2$

$$stst = tstt = ts$$

since

$$(st)^3 = e \Rightarrow sts = tst.$$

ts is a reduced expression of $stst$.

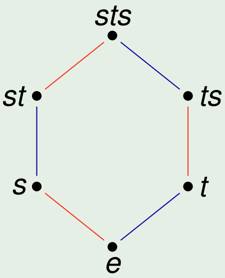
Weak order

- *(right) weak order* is the prefix order on elements:

$$u \leq_R v \iff \ell(v) = \ell(u) + \ell(u^{-1}v)$$

Example

Γ_{A_2} :



Inversion sets

$T = \cup wSw^{-1}$ is the set of *reflections*.

For $w \in W$, *(left) inversion set*:

$$N(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$$

Example

Γ_{A_2} : 

Recall $\ell(ts) = 2$. Then

$$N(ts) = \{t, tst\}$$

since

$$(t)(ts) = tts = s$$
$$(tst)(ts) = tstts = t$$

Generating inversion sets

For $w = s_1 \dots s_n$ a reduced expression:

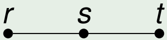
$$N(w) = \{s_1,\ s_1 s_2 s_1,\ s_1 s_2 s_3 s_2 s_1,\ \cdots,\ s_1 s_2 \dots s_n \dots s_2 s_1\}$$

Example

Γ_{A_2} : 

$$N(ts) = \{t, tst\}$$

Example

Γ_{A_3} : 

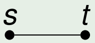
$$N(rstsr) = \{r, rsr, rstsr, rststsr = t, rstsrstsr = tst\}$$

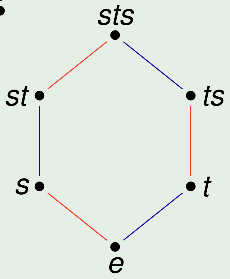
Weak order - Round 2

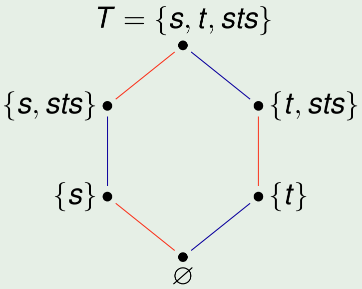
- The *(right) weak order* is the prefix order on elements:

$$u \leq_R v \iff \ell(v) = \ell(u) + \ell(u^{-1}v) \iff \mathbf{N(u)} \subseteq \mathbf{N(v)}$$

Example

Γ_{A_2} : 





(Semi)lattices

(W, S) a Coxeter system and $u, v \in W$.

- *Meet* (greatest lower bound) - $u \wedge_R v$
- *Join* (least upper bound) - $u \vee_R v$
- *Meet-semilattice* every two elements have a meet.
- *Join-semilattice* every two elements have a join.
- *Lattice* every two elements have a meet and a join.

Theorem (Björner [1984])

The weak order is a meet-lattice. If W is finite, it's a lattice.

Lattice

- *Meet* (greatest lower bound) - $u \wedge_R v$
- *Join* (least upper bound) - $u \vee_R v$
- *Meet-semilattice* every two elements have a meet.
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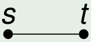
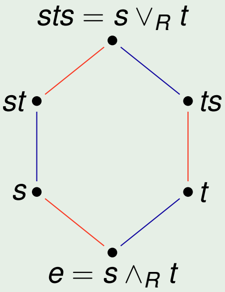
Example

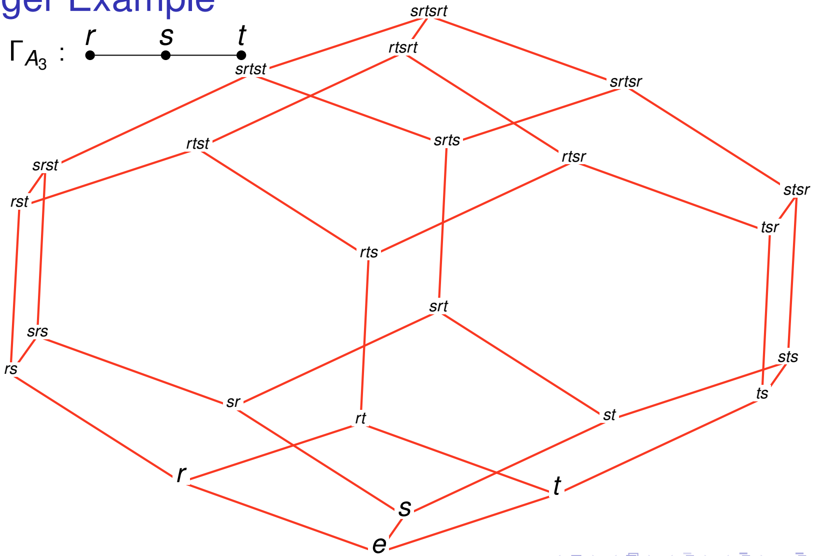
Γ_{A_2} :





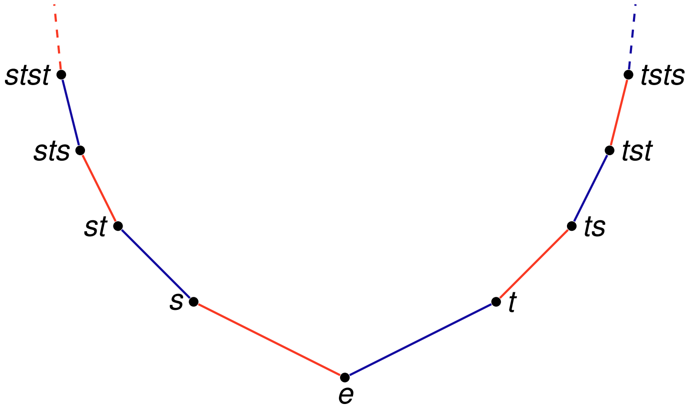
Larger Example

Γ_{A_3} :



Infinite example

$\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{\text{———}} t$

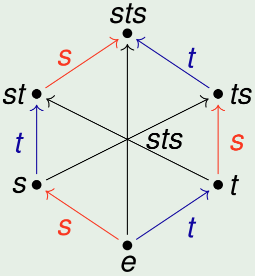


Bruhat graph Ω

- Vertices: Elements of W
- Edges: (u, v) if
  - $v = ut$ for some $t \in T$
  - $\ell(u) \leq \ell(v)$.

Example

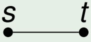
Γ_{A_2} :  .



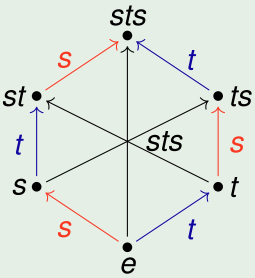
A-Path

For $A \subseteq T$, *A-path* a directed path in Ω following only edges labelled by elements in A.

Example

Γ_{A_2} : 

$A = \{s, sts\} \subseteq \{s, sts, t\} = T$



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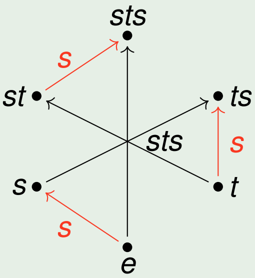

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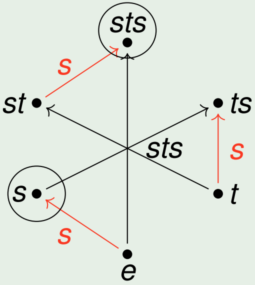

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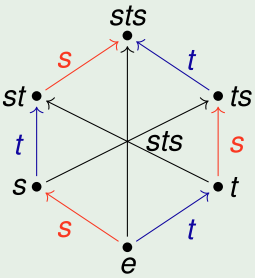
Example

Γ_{A_2} : s ——— t $\quad A = \{sts\} \subseteq \{s, sts, t\} = T$



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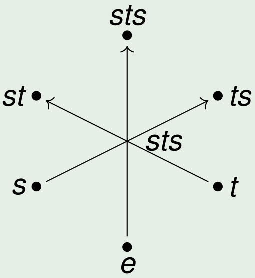

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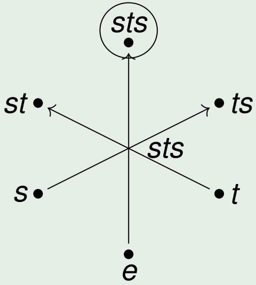

Bruhat preclosure

For $A \subseteq T$

$$\overline{A} = \{t \in T \mid \text{there exists an } A\text{-path from } e \text{ to } t\}$$

Example

Γ_{A_2} :

$$\begin{aligned} \overline{\{s, sts\}} &= \{s, sts\} \\ \overline{\{sts\}} &= \{sts\} \\ \overline{\{s, t\}} &= \{s, sts, t\} = T \end{aligned}$$

Bruhat preclosure

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Example

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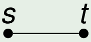

$$\overline{\{s, sts\}} = \{s, sts\} = N(st)$$
$$\overline{\{sts\}} = \{sts\}$$
$$\overline{\{s, t\}} = \{s, sts, t\} = T = N(sts) = N(s \vee_R t)$$

Dyer's conjecture

For $A \subseteq T$ and $u, v \in W$:

1. $\overline{A}$ is a closure operator.
2. $\overline{N(u) \cup N(v)} = N(u \vee_R v)$.

Dyer's conjecture

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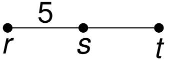
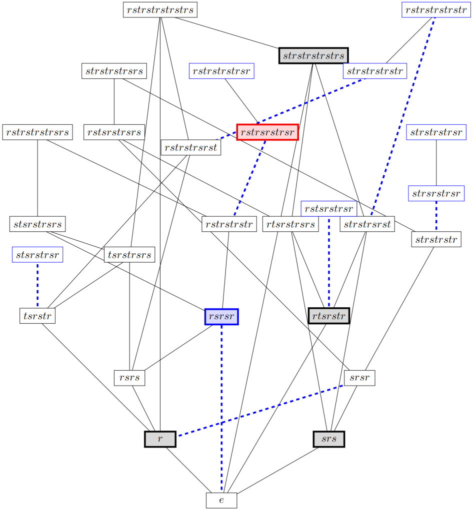
Counterexample

Γ_{H_3} :


$$A = \{r, srs, tsrsrst, strstrstrstrs\}$$
$$\overline{A} = A \cup \{srsrs\}$$
$$\overline{\overline{A}} = \overline{A} \cup \{rstrsrstrsr\}$$



Dyer's conjecture

For $A \subseteq T$ and $u, v \in W$:

1. ~~$\overline{A}$ is a closure operator.~~ Can we do something about this?
2. $\overline{N(u) \cup N(v)} = N(u \vee_R v)$.

Reflection order

reflection order $(T, \preccurlyeq)$: total order such that for every $r, s \in T$, $W' = \langle r, s \rangle$:

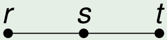
$$r \prec rsr \prec \cdots \prec srs \prec s$$

or

$$s \prec srs \prec \cdots \prec rsr \prec r$$

Example

Γ_{A_3} :



$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

$$r \preccurlyeq rsr \preccurlyeq rstsr \preccurlyeq s \preccurlyeq sts \preccurlyeq t$$

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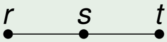
Γ_{A_3} : 

$$\langle s, t \rangle \Rightarrow \{s, t, sts\}$$

$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

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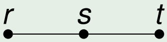

$$\langle r, s \rangle \Rightarrow \{r, s, rsr\}$$

$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

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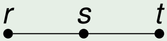

$$\langle r, sts \rangle \Rightarrow \{r, sts, rstsr\}$$

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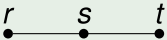


$$\langle rsr, t \rangle \Rightarrow \{rsr, t, rstsr\}$$

$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

$$r \preccurlyeq rsr \preccurlyeq rstsr \preccurlyeq s \preccurlyeq sts \preccurlyeq t$$

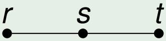
Initial section

Subset of T containing an "initial part" of *some* reflection order.

Finite = Inversion sets

Example

Γ_{A_3} :



$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

$$r \preccurlyeq rsr \preccurlyeq rstsr \preccurlyeq s \preccurlyeq sts \preccurlyeq t$$

$\{r, t\}$	$\{r, rsr, rstsr\}$	$\{r, t, rstsr, sts\}$	$\varnothing$	T
$N(rt)$	$N(rst)$	$N(rtsr)$	$N(e)$	$N(rstrst)$

Non-finite initial section: $\tilde{A}_1$

Subset of T containing an "initial part" of *some* reflection order.

Example

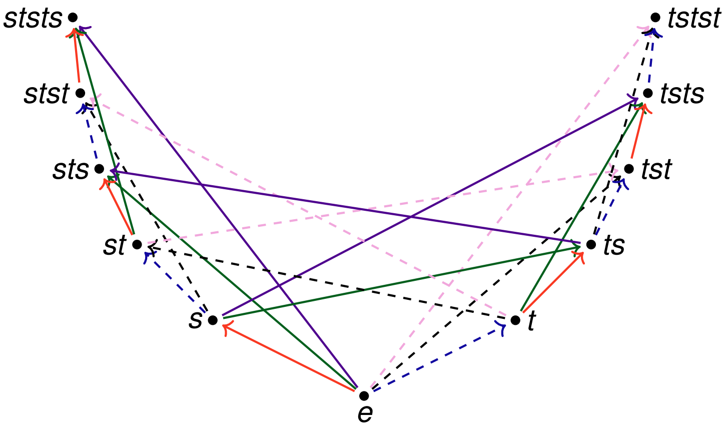
$\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{\bullet\!-\!-\!\bullet} t$

$$s \prec sts \prec ststs \prec \cdots \prec tstst \prec tst \prec t$$

Initial section: $\{w \in T \mid w \text{ begins with } s\}$

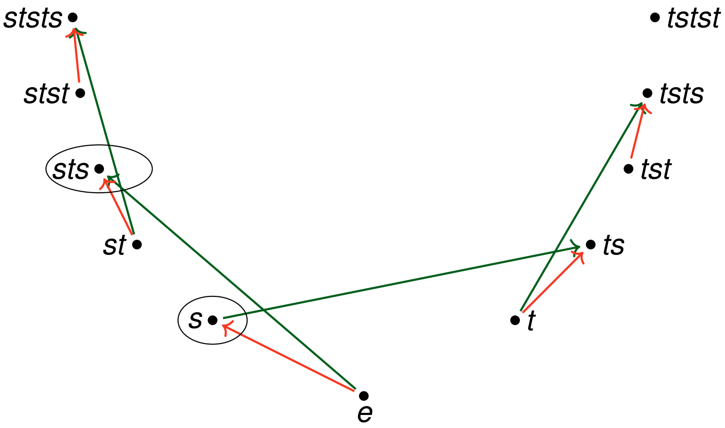
$\tilde{A}_1$ $\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{—} t$

s : red t : blue sts : green tst : black $ststs$: purple $tstst$: pink



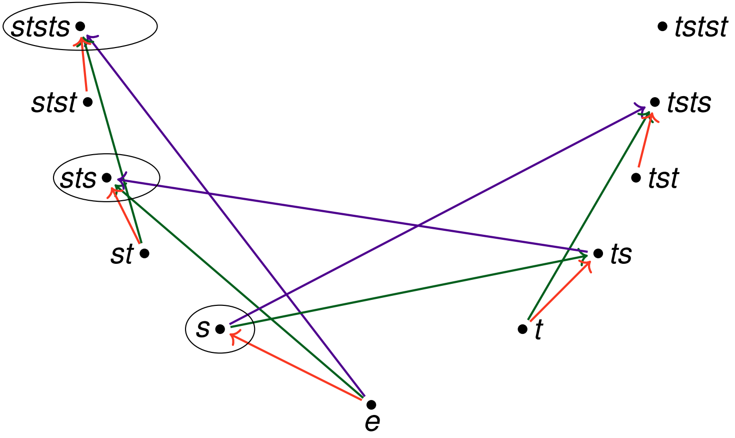
$\tilde{A}_1$ $\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{\text{———}} t$ $N(st) = \{s, sts\}$

s : red t : blue sts : green tst : black $ststs$: purple $tstst$: pink



$\tilde{A}_1$ $\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{—} t$ $\{w \in T \mid w \text{ begins with } s\}$

s : red t : blue sts : green tst : black $ststs$: purple $tstst$: pink

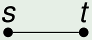


Initial sections

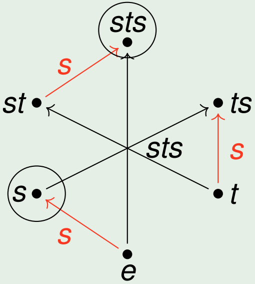
Theorem (D. 2025+)

If $A \subseteq T$ is an initial section, then $\overline{A} = A$.

Example

Γ_{A_2} : 

$A = N(st) = \{s, sts\} \subseteq \{s, sts, t\} = T$.



Finite type preclosure

Z a set, $\mathrm{cl} : \mathcal{P}(Z) \to \mathcal{P}(Z)$ a preclosure.
cl is of *finite type* if

$$\mathrm{cl}(X) = \cup \{\mathrm{cl}(Y) \mid Y \subseteq X, |Y| \in \mathbb{N}\}$$

Note: $\mathrm{cl}^\infty(X) = \cup_{i \in \mathbb{N}} \mathrm{cl}^i(X)$ is a closure.

Theorem (D. 2025+)

Bruhat preclosure is of finite type.

Note: $\overline{A}^i$ is Bruhat preclosure performed i times:

$$\overline{A}^2 = \overline{\overline{A}} \quad \overline{A}^3 = \overline{\overline{\overline{A}}}$$

Corollary

*For *any* initial section $A \subseteq T$, $\overline{A}^\infty$ is a closure.*

Dyer's conjecture

For $A \subseteq T$ and $u, v \in W$:

1. ~~$\overline{A}$ is a closure operator.~~ Kinda fixed - Initial sections + Infinite Bruhat closure
2. $\overline{N(u) \cup N(v)} = N(u \vee_R v)$.

Main theorem

Theorem (D. 2025+)

Let W be an arbitrary Coxeter group. For $u, v \in W$ such that $u \vee_R v$ exists, then

$$\overline{N(u) \cup N(v)}^{\infty} = N(u \vee_R v)$$

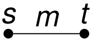
Corollary (D. 2025+)

For W a finite Coxeter group, then for all $u, v \in W$

$$\overline{N(u) \cup N(v)}^{i} = N(u \vee_R v)$$

where $i \leq |T \backslash (N(u) \cup N(v))|$.

Progress - Dihedral

$\Gamma_{I_2(m)}$: 

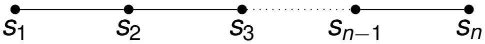
Theorem (Biagioli, Perrone 2025+, D. 2025+)

For W a dihedral group, for all $u, v \in W$,

$$\overline{N(u) \cup N(v)} = N(u \vee_R v).$$

Progress - Symmetric group - type A_n

Γ_{A_n} :



Theorem (Biagioli, Perrone 2025+)

For W a type A_n Coxeter group, for all $u, v \in W$

$$\overline{N(u) \cup N(v)} = N(u \vee_R v).$$

Theorem (D. 2025+)

For W a type A_n Coxeter group and $A \subseteq T$, $\overline{A}$ is a closure.

Progress - Other types

Finite types:

B_n Recently proved by Biagioli, Perrone 2026+

D_n Still open (Checked for $n \leq 5$)

F_4 Computer check by Biagioli, Perrone 2026+

H_3 Computer check by Biagioli, Perrone 2026+

H_4 Computer check by Dermenjian 2026+

E Still open (Check for E_6 in progress)

Infinite types:

- Everything open

Thank you