The strong exchange property for Coxeter matroids

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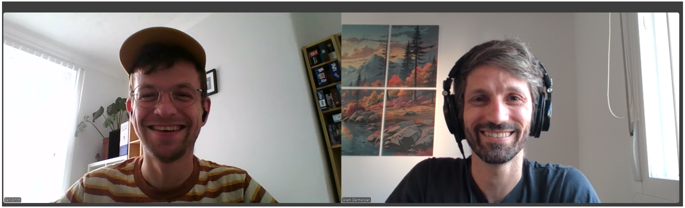
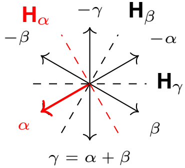
Collaborators



Roadmap

Root systems

- $(V, \langle\cdot,\cdot\rangle)$ a real Euclidean vector space
- For $\alpha \in V \backslash \{0\}$, let s_α be its *reflection* with hyperplane $H_\alpha = \{x \in V \mid \langle x, \alpha\rangle = 0\}$.
- A *root system* is $\Phi \subset V$ such that for all $\alpha, \beta \in \Phi$.
  - $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$,
  - $s_\alpha(\Phi) = \Phi$,
  - $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$
- $\Delta \subset \Phi$ a choice of linearly independent *simple roots*.



Weyl groups

- For a root system Φ and a fixed Δ the *Weyl group* of Φ is

$$W(\Phi) := \langle s_\alpha \mid \alpha \in \Delta \rangle .$$

- Finite types: A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2
- For $J \subset \Delta$, its *parabolic subgroup* $W_J \subset W(\Phi)$

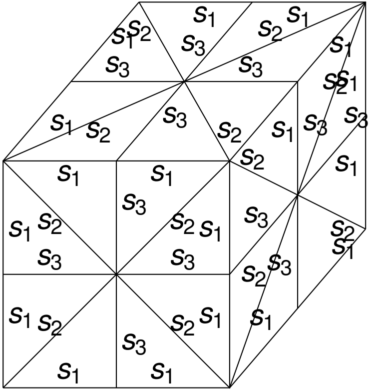
$$W_J := \langle s_\beta \mid \beta \in J \rangle .$$

- Quotients $W^J = W/W_J$
- *Parabolic cosets:* wW_J some representative for $w \in W^J$.

Geometry of cosets

For $J \subseteq \Delta$, define x_J such that: $\langle x_J, \alpha \rangle \begin{cases} = 0 & \alpha \in J \\ < 0 & \alpha \notin J \end{cases}$

$$\{\textbf{Cosets: } wW_J\} \longleftrightarrow \{\textbf{Points: } w(x_J)\}$$



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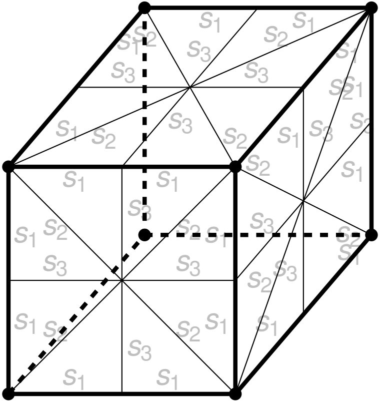

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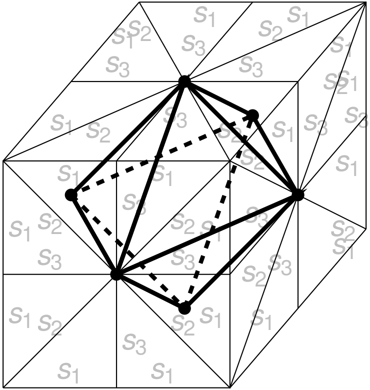
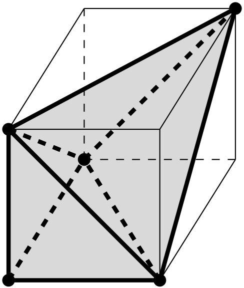


Coxeter matroids

Fix a root system Φ and a subset $J \subseteq \Delta$.

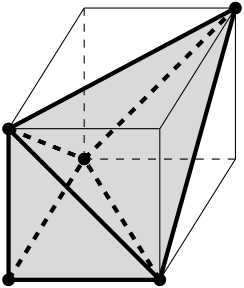
A subset $M \subseteq W(\Phi)^J$ is a *Coxeter matroid* if every edge of $\operatorname{conv}\{w(x_J) \mid w \in M\}$ is parallel to some $\alpha \in \Phi$.



Strong Coxeter matroids

A Coxeter matroid $M \subseteq W^J$ is a *strong* if it satisfies the *strong exchange property* for Coxeter matroids:

$$\forall v, w \in M, \exists H_\alpha \text{ separating } v, w \text{ such that } s_\alpha v, s_\alpha w \in M.$$



Question

How do we know if a given Coxeter matroid is strong by using only the polytope?

Grassmannians

Grassmannian: $\mathrm{Gr}(k,n) := \{\ k\text{-dim subspaces of } \mathbb{C}^n\ \}$

$$\mathrm{Gr}(k,n) \hookrightarrow \mathrm{Proj}\left(\bigwedge^k \mathbb{C}^n\right) \cong \mathbb{P}^{\binom{n}{k}-1}$$

$\mathrm{Gr}(k,n)$ cut out by the *Plücker equations:*

$$I_{k,n} := \left\langle \sum_{j\in J\setminus I} (-1)^{\bullet} \cdot X_{I+j} \cdot X_{J-j} \;\middle|\; I \in \binom{[n]}{k-1},\ J \in \binom{[n]}{k+1} \right\rangle .$$

Example ($k=2, n=4$)

$\mathrm{Gr}(2,4) = V(I_{2,4}) \subseteq \mathbb{P}^5$ where

$$I_{2,4} := \langle X_{12}\cdot X_{34} - X_{13}\cdot X_{24} + X_{14}\cdot X_{23}\rangle$$
$$\subseteq \mathbb{C}[X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}].$$

Tropicalization

Boolean semiring $\mathbb{B} := (\{0,1\}, \oplus, \cdot)$ with $1 \oplus 1 = 1$.
A *tropical equation* f with support $A \subset \mathbb{N}^n$ is

$$f := \bigoplus_{a \in A} X^a \in \mathbb{B}[X_1, \ldots, X_n], \quad X^a := X_1^{a_1} \cdots X_n^{a_n}$$

$$\begin{aligned} \mathrm{trop}\colon \mathbb{C}[X] &\longrightarrow \mathbb{B}[X] \\ \sum_{a \in A} z_a X^a &\longmapsto \bigoplus_{a \in A} X^a \end{aligned}$$

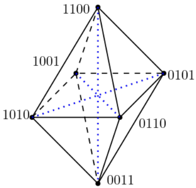
We say $v \in \mathbb{B}^n$ *satisfies* f if $|\{a \in A \mid v^a = 1\}| \neq 1$.

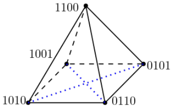
Tropicalization example

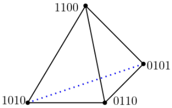
- $1 \oplus 1 = 1$.
- $\mathrm{trop}\left(\sum_{a \in A} z_a X^a\right) = \bigoplus_{a \in A} X^a$.
- $v \in \mathbb{B}^n$ *satisfies* f if $|\{a \in A \mid v^a = 1\}| \neq 1$.

Example ($n = 4, k = 2$)

$$\mathrm{trop}\left(X_{12} \cdot X_{34} - X_{13} \cdot X_{24} + X_{14} \cdot X_{23}\right)$$
$$= X_{12} \cdot X_{34} \oplus X_{13} \cdot X_{24} \oplus X_{14} \cdot X_{23}$$







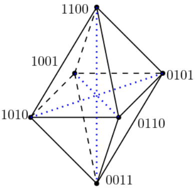
$$v_1 \to 1 \cdot 0 \oplus 1 \cdot 1 \oplus 1 \cdot 1 = 0 \oplus 1 \oplus 1 \Rightarrow |\{a \in A \mid v_1^a = 1\}| = 2$$
$$v_2 \to 1 \cdot 0 \oplus 1 \cdot 1 \oplus 0 \cdot 1 = 0 \oplus 1 \oplus 0 \Rightarrow |\{a \in A \mid v_2^a = 1\}| = 1$$
$$v_3 \to 1 \cdot 1 \oplus 1 \cdot 1 \oplus 1 \cdot 1 = 1 \oplus 1 \oplus 1 \Rightarrow |\{a \in A \mid v_3^a = 1\}| = 3$$

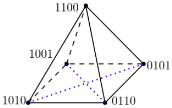
Tropicalization example

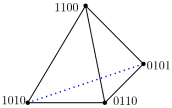
- $1 \oplus 1 = 1$.
- $\text{trop}\left(\sum_{a \in A} z_a X^a\right) = \bigoplus_{a \in A} X^a$.
- $v \in \mathbb{B}^n$ *satisfies* f if $|\{a \in A \mid v^a = 1\}| \neq 1$.

Example ($n = 4, k = 2$)

$$\text{trop}\left(X_{12} \cdot X_{34} - X_{13} \cdot X_{24} + X_{14} \cdot X_{23}\right)$$
$$= X_{12} \cdot X_{34} \oplus X_{13} \cdot X_{24} \oplus X_{14} \cdot X_{23}$$







$$v_1 \to 1 \cdot 0 \oplus 1 \cdot 1 \oplus 1 \cdot 1 = 0 \oplus 1 \oplus 1 \Rightarrow |\{a \in A \mid v_1^a = 1\}| = 2 \quad \textit{Yes!}$$
$$v_2 \to 1 \cdot 0 \oplus 1 \cdot 1 \oplus 0 \cdot 1 = 0 \oplus 1 \oplus 0 \Rightarrow |\{a \in A \mid v_2^a = 1\}| = 1 \quad \textit{No!}$$
$$v_3 \to 1 \cdot 1 \oplus 1 \cdot 1 \oplus 1 \cdot 1 = 1 \oplus 1 \oplus 1 \Rightarrow |\{a \in A \mid v_3^a = 1\}| = 3 \quad \textit{Yes!}$$

The combinatorial Grassmannian

Theorem (Speyer '08)

$M \subseteq \binom{[n]}{k}$ is a (strong) matroid if and only if v^M satisfies the tropicalized Plücker equations:

$$\operatorname{trop}(\mathcal{P}_{I,J}) := \bigoplus_{j \in J \setminus I} X_{I+j} \cdot X_{J-j} \in \mathbb{B}[X] \quad \forall I \in \binom{[n]}{k-1}, J \in \binom{[n]}{k+1}.$$

Lie theory

Find tropical equations that cut out a combinatorial Grassmannian for Coxeter matroids.

- Φ root system,
- $\mathbb{G}$ its complex simply connected Lie group,
- $\mathfrak{g}$ its Lie algebra.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \,, \quad \Phi \subset \mathfrak{h}^*$$

A $\mathfrak{g}$-module V decomposes into weight spaces $V = \bigoplus V_\mu$ where

$$V_\mu := \{ v \in V \mid h \cdot v = \mu(h) v \; \forall h \in \mathfrak{h} \} \,, \quad \mu \in \mathfrak{h}^* \,.$$

V a representation of $\mathfrak{g}$ $\Rightarrow$ weights are invariant under $W(\Phi)$.

Grassmannians as group orbits

- Simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$,
- Maximal parabolic subgroups $W_{\Delta \setminus \alpha_i} \leq W(\Phi)$,
- Fundamental representations V_{λ_i} of $\mathfrak{g}$,
- Maximal parabolic subgroups $\mathbb{P}_{\alpha_i} = \mathrm{stab}_{\mathbb{G}}(v_{\lambda_i}) \leq \mathbb{G}$.

$$\mathbb{G}/\mathbb{P}_{\alpha_i} \hookrightarrow \mathrm{Proj}(V_{\lambda_i}) \qquad g \mapsto g \cdot v_{\lambda_i}$$

Example ($\Phi = A_{n-1}, \mathbb{G} = SL_n(\mathbb{C})$)

The simple root α_k has fundamental representation $V_{\lambda_k} = \bigwedge^k \mathbb{C}^n$.

$$\mathrm{Gr}(k, n) \cong SL_n(\mathbb{C}) / \mathrm{stab}_{SL_n(\mathbb{C})}(U) \hookrightarrow \mathrm{Proj}\left(\bigwedge^k \mathbb{C}^n\right)$$

where $U \subseteq \mathbb{C}^n$ standard k-dim linear space.

Quadric equations

$$\mathbb{G}/\mathbb{P}_{\alpha_i} \hookrightarrow \mathrm{Proj}(V_{\lambda_i})$$

How do we characterise $\mathbb{G}/\mathbb{P}_{\alpha_i}$ via equations?

Theorem (Lichtenstein 1982)

The elements of $\mathrm{Sym}^2(V_\lambda^\vee)$ *satisfying*

$$\Omega(u \otimes v) - \langle \lambda, 2\lambda + 2\rho \rangle (u \otimes v) = 0$$

gives a system of quadratics cutting out $\mathbb{G}/\mathbb{P}$ *inside* $\mathrm{Proj}(V_\lambda)$*.*

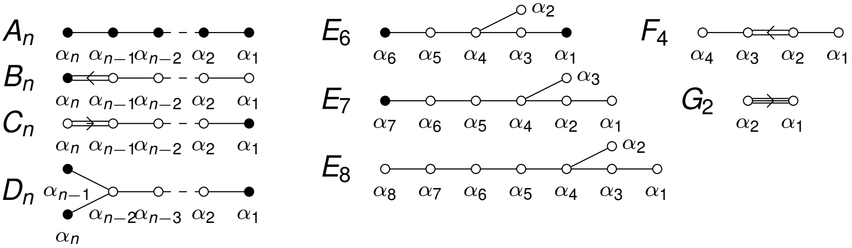
This gives ‘equations’, but to get **equations**, need a canonical basis for V_λ.

Miniscule

Fundamental rep. V_{λ_i} is *minuscule* if $W(\Phi)$ acts transitively on the weights.

$\rightsquigarrow V_{\lambda_i} = \bigoplus V_\mu$ decomposes into one-dimensional weight spaces.

We call $\mathbb{P}_{\alpha_i} \leq \mathbb{G}$ minuscule if V_{λ_i} minuscule.



Strong Coxeter matroids characterization

Theorem (Calvert, D., Fink, Smith '25+)

Let Φ be a root system with $\mathbb{G}$ simply connected complex Lie group and $\mathbb{P}_\alpha$ some minuscule parabolic subgroup. There exists a set of quadrics F cutting out $\mathbb{G}/\mathbb{P}_\alpha \subseteq \mathrm{Proj}(V_\lambda)$ such that:

$$M \subseteq W^{\Delta\setminus\alpha} \text{ a strong Coxeter matroid } \iff v^M \text{ satisfies } \mathrm{trop}(F).$$

How to find F:

- Decompose $\mathrm{Sym}^2(V_\lambda) = \bigoplus V_\mu$,
- Each $\mu \neq 2\lambda$ gives equations $\omega = 0$ for each basis vector $\omega \in V_\mu$,
- Rewrite ω in the V_λ basis to get an **equation.**

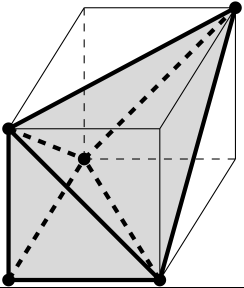
Example: $\Phi = B_3$ $\quad \mathbb{G}/\mathbb{P}_{\alpha_3} \hookrightarrow \mathrm{Proj}(\mathcal{S}(3))$

- $\mathcal{S}(3)$ spin module: 2^3-dim with weights $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$.
- $\mathbb{G}/\mathbb{P}_{e_3}$ cut out by a single equation f with tropicalization:

$$\mathrm{trop}(f) := x_1x_2 \oplus x_3x_4 \oplus x_5x_6 \oplus x_7x_8 \in \mathbb{B}[X]$$

$$M \subseteq W^{\Delta\setminus\alpha_3} \text{ strong} \iff v^M \text{ satisfies } \mathrm{trop}(f)$$

$$\iff \mathrm{conv}\left\{w(x_J) \mid w \in M\right\} \text{ no unique antipode.}$$



Future Work

- How do we consider non-minuscule representations/parabolics?
- Any hope of quantifying non-strong Coxeter matroids?

Thanks