Enumerating Weyl Cones of Shi Arrangements

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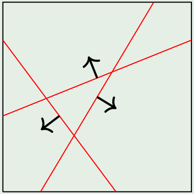
Hyperplane Arrangements

- $(V, \langle\cdot,\cdot\rangle)$ - n-dim real Euclidean vector space.
- A *hyperplane H* is a codim 1 subspace of V.
- A *(hyperplane) arrangement* is a *finite* collection of hyperplanes.

Example

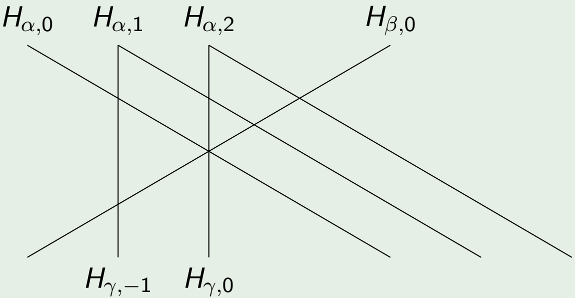


Hyperplanes and vectors

For $\alpha \in \mathbb{R}^n$ a vector.

- $H_{\alpha,k} = \{v \in \mathbb{R}^n \mid \langle \alpha, v \rangle = k\}$ - hyperplane.
- $H_\alpha = H_{\alpha,0}$ - central hyperplane.
- s_α - reflection fixing H_α pointwise.

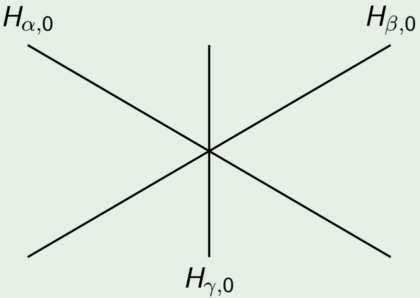
Example



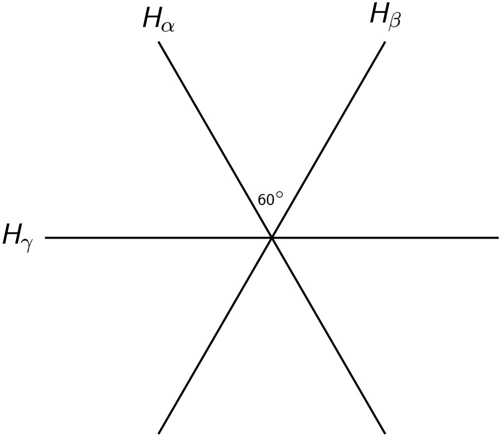
Central arrangements

A *central arrangement* is a a hyperplane arrangement with only central hyperplanes.

Example



A “nice” central hyperplane arrangement



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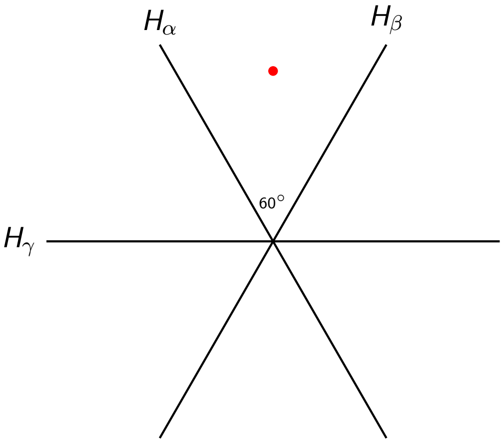

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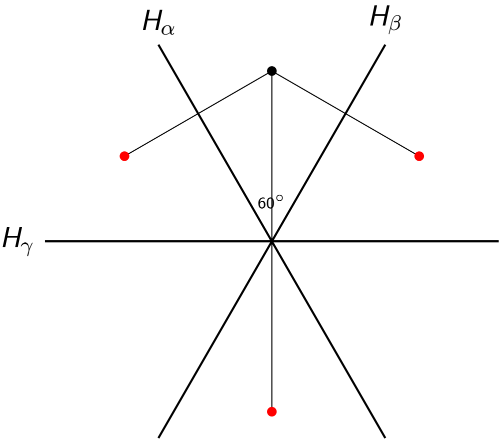

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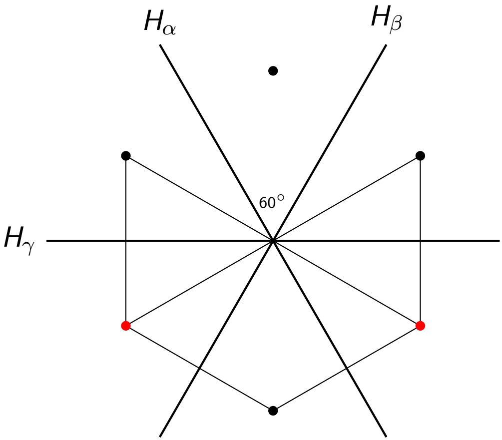

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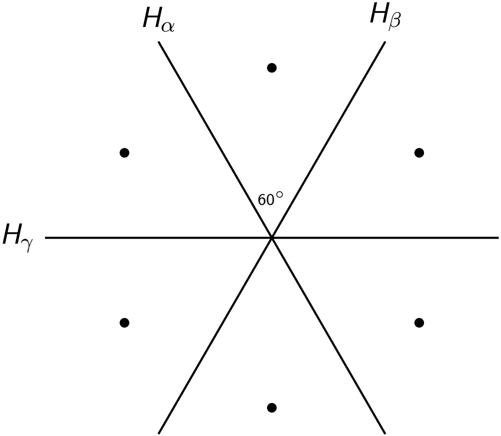

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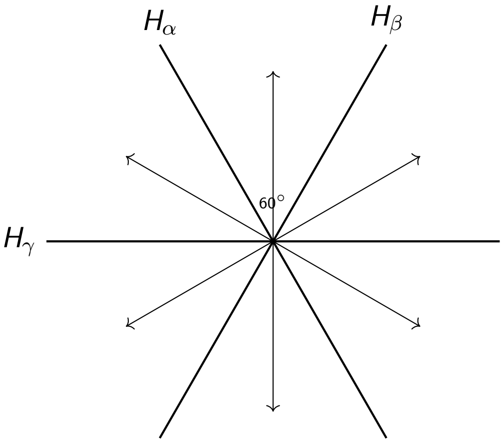

Root Systems

Definition

A *root system* Φ is (finite) collection of nonzero vectors satisfying:

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for every $\alpha \in \Phi$.
2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
3. $\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The $\alpha \in \Phi$ are called *roots*.

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- Φ^+ - Positive roots
- Φ^- - Negative roots
- Δ - Simple roots
- $W = \langle S \rangle$, $S = \{s_\alpha \mid \alpha \in \Delta\}$ - Weyl group.

Coxeter and Shi Arrangements

Definitions

A *Coxeter arrangement* is the arrangement for a root system Φ:

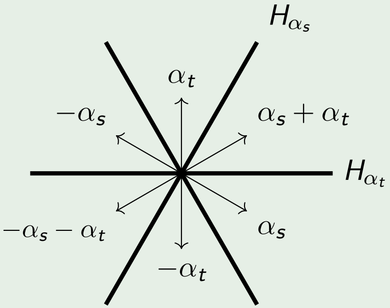
$$\mathcal{A}(\Phi) = \{ H_\alpha \mid \alpha \in \Phi^+ \} .$$

A *Shi arrangement* is the Coxeter arrangement together with a positive unit translate of each hyperplane:

$$\mathrm{Shi}(\Phi) = \{ H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \{0,1\} \}$$

A_2 example

Example (Coxeter Arrangement)



A_2 example

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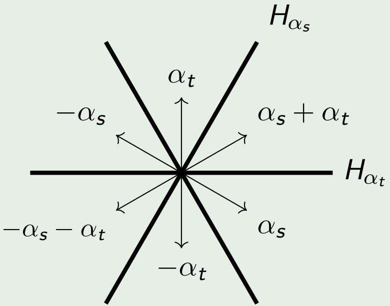
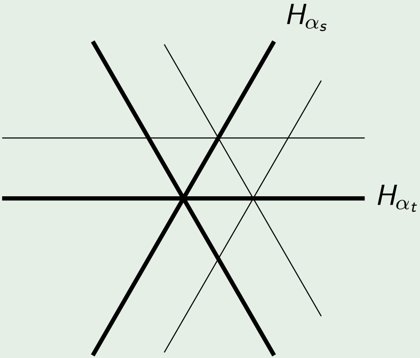


- $\Phi^+ = \{\alpha_s, \alpha_t, \alpha_s + \alpha_t\}$
- $\Phi^- = \{-\alpha_s, -\alpha_t, -\alpha_s - \alpha_t\}$
- $\Delta = \{\alpha_s, \alpha_t\}$

A_2 example

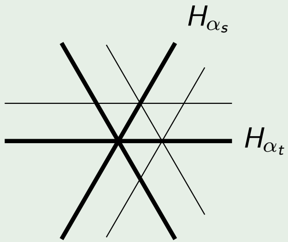
Example (Shi Arrangement)



Regions

A *region* is a (open) connected component of the vector space with the hyperplanes removed.

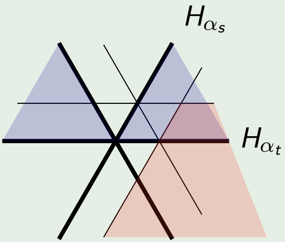
Example (Shi Arrangement)



Cone

A *cone* is an intersection of (open) half-spaces of (some) hyperplanes.

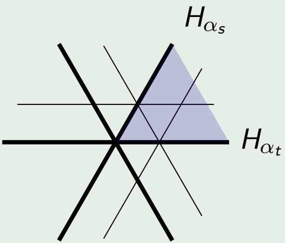
Example (Shi Arrangement)



Weyl cone

For Shi(Φ), the regions of the Coxeter subarrangement are in bijection with the elements of W. These regions define cones called *Weyl cones*. The cone associated to the identity is commonly referred to as the *dominant cone*.

Example (Shi Arrangement)



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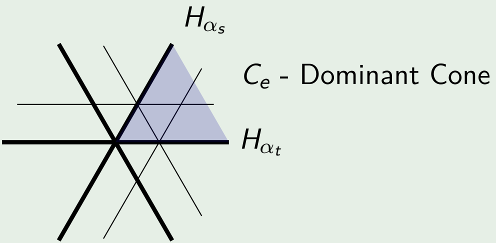

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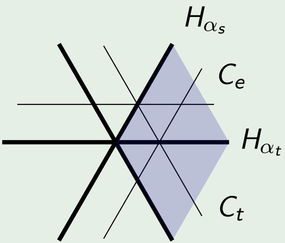

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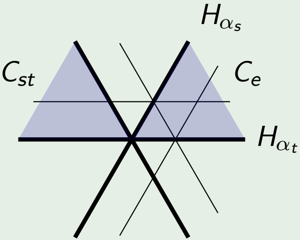


Question:

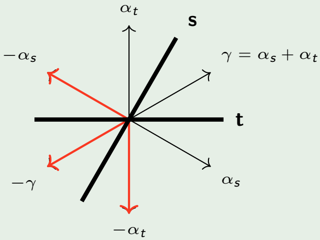
How many regions are in each Weyl cone?

Inversion Sets

The *(left) inversion sets* is the set

$$N(w) = \Phi^+ \cap w(\Phi^-).$$

Example



$$\begin{aligned} N(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\} \end{aligned}$$

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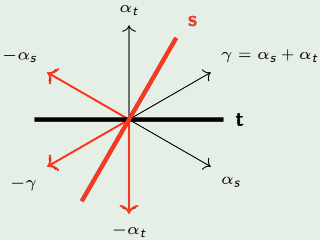

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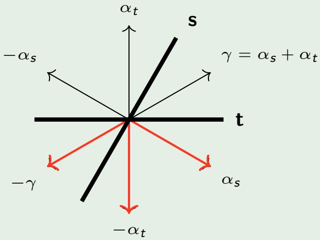

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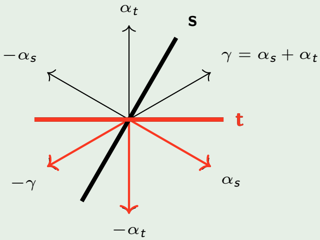

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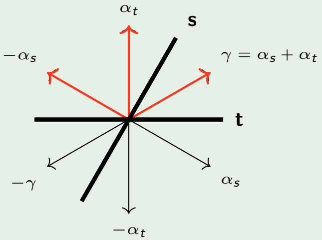

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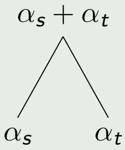
Root poset

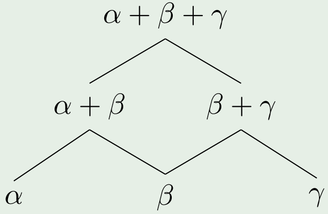
Definition

The *root poset* $(\Phi^+, \leq)$ is the poset where

$$\alpha < \beta \iff \beta - \alpha \in \mathbb{N}\Delta$$

Example

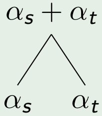




Antichain

An *antichain* in a poset is a set of pairwise incomparible elements.

Example

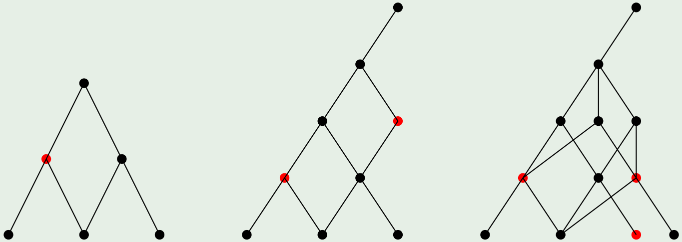


There are 5 antichains:

$$\varnothing, \{\alpha_s\}, \{\alpha_t\}, \{\alpha_s + \alpha_t\}, \{\alpha_s, \alpha_t\}$$

Root Posets

Example (A_3, B_3, D_4)

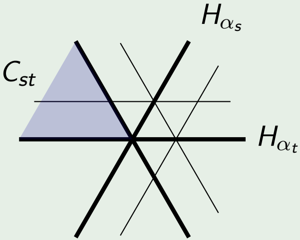


Number of regions using antichains

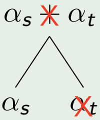
Theorem (Armstrong, Reiner, Rhoades 2015)

The number of regions in a Weyl cone C_w is equal to the number of antichains in the subposet of the root poset $(\Phi^+, \leq)$ restricted to $\Phi^+ \setminus N(w^{-1})$.

Example (A_2 Shi Arrangement)



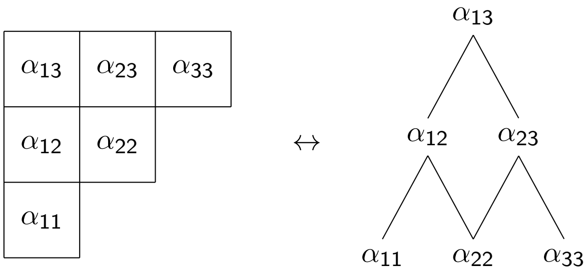
$N(ts) = \{\alpha_t, \alpha_s + \alpha_t\}$



2 antichains: $\varnothing$, $\{\alpha_s\}$

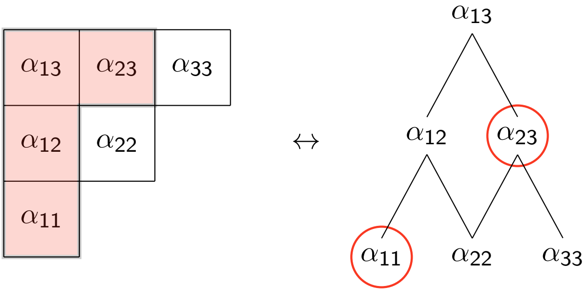
Diagrams (type A)

Shorthand: $\alpha_{ij} = \sum_{k=i}^{j} \alpha_k$



Subdiagrams

A *subdiagram* is a set B of boxes such that if $b \in B$ then every box above and to the left are also in B.



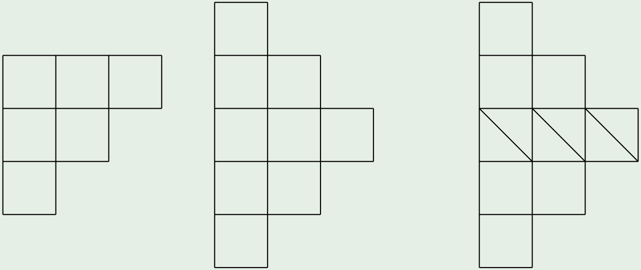
If a box is in the bottom right corner of the subdiagram, it is in antichain.

Subdiagrams

Theorem (Shi 1995)

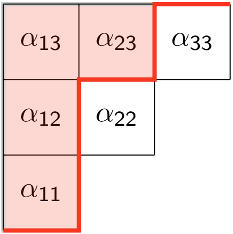
Let Λ be the diagram associated to a Coxeter group W with root system Φ. Then there is a bijection between number of subdiagrams of Λ and antichains in $(\Phi^+, \leq)$.

Example (A_3, B_3, D_4)

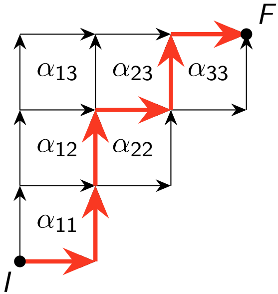


Diagrams to Digraphs - Type A

Shorthand: $\alpha_{ij} = \sum_{k=i}^{j} \alpha_k$

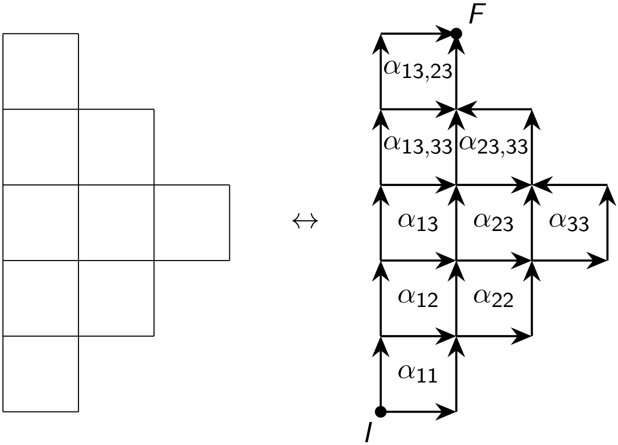


$\leftrightarrow$

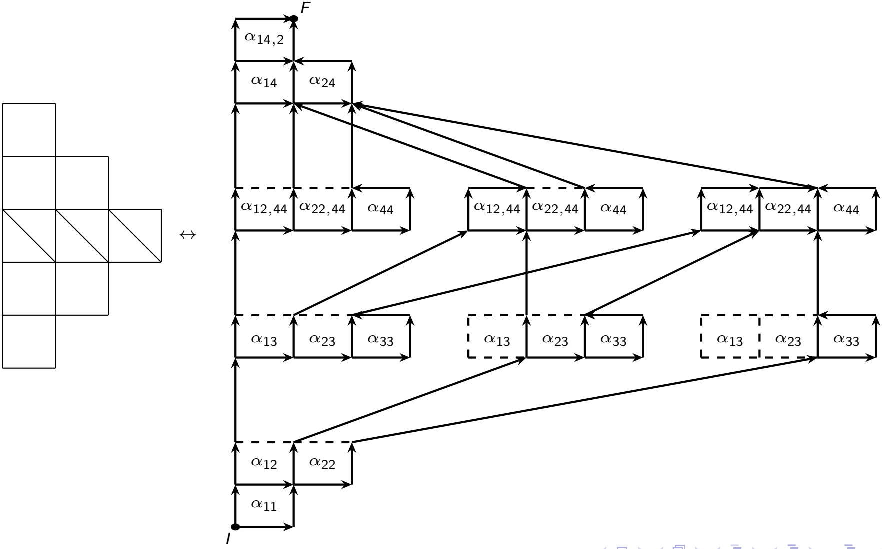


Diagrams to Digraphs - Type B

Shorthand: $\alpha_{ij,k\ell} = \alpha_{ij} + \alpha_{k\ell}$



Diagrams to Digraphs - Type D

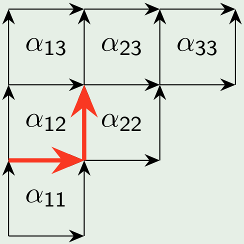


Corners

For each $\alpha \in \Phi^+$ we let Π_α be the set of subpaths of Γ which go under and to the right of α.

Example

$\Pi_{\alpha_{12}}$ is associated to the following subpath.



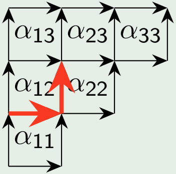
Digraph solution

Let $$\Pi_w = \bigcup_{\alpha \in N(w^{-1})} \Pi_\alpha$$

Theorem (D., Tzanaki 2023)

Let Γ be the digraph associated to W with root system Φ. There is a bijection between paths in Γ which don't contain subpaths in Π_w and antichains in the root poset $(\Phi^+, \leq)$ restricted to $\Phi^+ \setminus N(w^{-1})$.

Example



But..

How does this help?

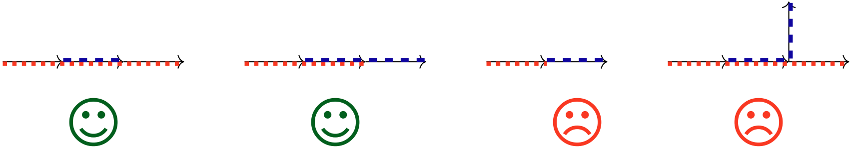
Digraphs - Notation

- Directed Graph (Digraph): Γ
- Path: $\pi = (v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n)$
- Let $I_\pi = v_1$ and $F_\pi = v_n$ (Initial/Final vertices)
- Γ is *acyclic* if there are no paths such that $I_\pi = F_\pi$.

Overlapping paths

Two paths π and $\pi' = (u_1, f_1, \ldots, f_{m-1}, u_m)$ *overlap* if:

- π is a subpath of π', or
- there exists some $i \in [n-1]$ such that for all $j \in [n-i]$, then $e_{i+j-1} = f_j$ (the final i *edges* in π coincide with the first i edges of π').



Non-overlapping collections

A collection of paths Π is *non-overlapping* if there does not exist any $\pi, \pi' \in \Pi$ such that π overlaps π'.

Let $\gamma(v \to v')$ be the number of paths from v to v'.

Number of paths

Collection of non-overlapping: pair-wise non-overlapping

$\gamma(v \to v') = \#$ paths from v to v'.

Theorem (D., Tzanaki 2023)

Let I and F be two arbitrary vertices in an acyclic digraph Γ. Let Π be a collection of non-overlapping paths. Then the number of paths from I to F which do not contain a path in Π as a subpath is equal to:

$$\det \begin{pmatrix} 1 & \gamma(F_2 \to I_1) & \cdots & \gamma(F_n \to I_1) & \gamma(I \to I_1) \\ \gamma(F_1 \to I_2) & 1 & \cdots & \gamma(F_n \to I_2) & \gamma(I \to I_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma(F_1 \to I_n) & \gamma(F_2 \to I_n) & \cdots & 1 & \gamma(I \to I_n) \\ \gamma(F_1 \to F) & \gamma(F_2 \to F) & \cdots & \gamma(F_n \to F) & \gamma(I \to F) \end{pmatrix}$$

Path enumeration

Theorem (André 1887)

Let Γ be the infinite digraph of $\mathbb{Z}^2$ with vertical edges pointing north and horizontal edges pointing east. Label every vertex of Γ by its respective coordinates in $\mathbb{Z}^2$. Then the number of paths from (x_1, y_1) to (x_2, y_2) weakly above the $x = y$ diagonal is given by:
If $x_1 \leq x_2$ and $y_1 \leq y_2$:

$$\binom{x_2 + y_2 - x_1 - y_1}{y_2 - y_1} - \binom{x_2 + y_2 - x_1 - y_1}{y_2 - x_1 + 1}$$

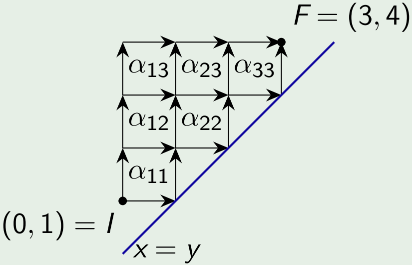
and 0 otherwise.

Type A

Let Γ be the infinite digraph of $\mathbb{Z}^2$.

- $I = (0,1)$ and $F = (n, n+1)$.
- $\alpha_{ij} = \sum_{k=i}^{j} \alpha_k \in \Phi$, $\Rightarrow \pi_{ij} : (i-1, j) \to (i, j) \to (i, j+1)$.

Example



Type A Determinant

Theorem (D., Tzanaki 2023)

For W type A and inversion set $N(w^{-1}) = \{\alpha_{i_1 j_1}, \ldots, \alpha_{i_k j_k}\}$ for $w \in W$. The number of regions in C_w is:

$$\det \begin{pmatrix} 1 & \gamma\left(v_2^{tr} \to v_1^{bl}\right) & \cdots & \gamma\left(v_k^{tr} \to v_1^{bl}\right) & \gamma\left(I \to v_1^{bl}\right) \\ \gamma\left(v_1^{tr} \to v_2^{bl}\right) & 1 & \cdots & \gamma\left(v_k^{tr} \to v_2^{bl}\right) & \gamma\left(I \to v_2^{bl}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma\left(v_1^{tr} \to v_k^{bl}\right) & \gamma\left(v_2^{tr} \to v_k^{bl}\right) & \cdots & 1 & \gamma\left(I \to v_k^{bl}\right) \\ \gamma\left(v_1^{tr} \to F\right) & \gamma\left(v_2^{tr} \to F\right) & \cdots & \gamma\left(v_k^{tr} \to F\right) & \gamma\left(I \to F\right) \end{pmatrix},$$

where $I = (0,1)$, $F = (n, n+1)$, $v_\ell^{tr} = (i_\ell, j_\ell + 1)$ and $v_\ell^{bl} = (i_\ell - 1, j_\ell)$.

A_5 example

Let W be the A_5 Coxeter arrangement and $w = s_5 s_2 s_4 s_3 s_1$. Then

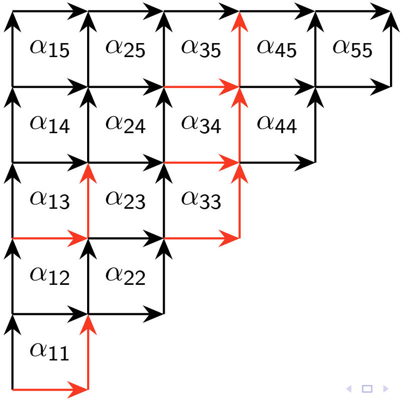
$$N(w^{-1}) = \{\alpha_{11}, \alpha_{33}, \alpha_{34}, \alpha_{13}, \alpha_{35}\}$$

$$\begin{aligned}
\alpha_{11} &\leftrightarrow (0,1) \to (1,1) \to (1,2) \\
\alpha_{33} &\leftrightarrow (2,3) \to (3,3) \to (3,4) \\
\alpha_{34} = \alpha_3 + \alpha_4 &\leftrightarrow (2,4) \to (3,4) \to (3,5) \\
\alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3 &\leftrightarrow (0,3) \to (1,3) \to (1,4) \\
\alpha_{35} = \alpha_3 + \alpha_4 + \alpha_5 &\leftrightarrow (2,5) \to (3,5) \to (3,6)
\end{aligned}$$

A_5 example cont.

$$N(w^{-1}) = \{\alpha_{11}, \alpha_{33}, \alpha_{34}, \alpha_{13}, \alpha_{35}\}$$

$$\Pi_w = \left\{\pi_{ij} \;\middle|\; \alpha_{ij} \in N(w^{-1})\right\}$$



A_5 example cont.

The number of regions in C_w is equal to

$$\det \begin{pmatrix} 1 & \gamma((3,4)\to(0,1)) & \gamma((3,5)\to(0,1)) & \gamma((1,4)\to(0,1)) & \gamma((3,6)\to(0,1)) & \gamma((0,1)\to(0,1)) \\ \gamma((1,2)\to(2,3)) & 1 & \gamma((3,5)\to(2,3)) & \gamma((1,4)\to(2,3)) & \gamma((3,6)\to(2,3)) & \gamma((0,1)\to(2,3)) \\ \gamma((1,2)\to(2,4)) & \gamma((3,4)\to(2,4)) & 1 & \gamma((1,4)\to(2,4)) & \gamma((3,6)\to(2,4)) & \gamma((0,1)\to(2,4)) \\ \gamma((1,2)\to(0,3)) & \gamma((3,4)\to(0,3)) & \gamma((3,5)\to(0,3)) & 1 & \gamma((3,6)\to(0,3)) & \gamma((0,1)\to(0,3)) \\ \gamma((1,2)\to(2,5)) & \gamma((3,4)\to(2,5)) & \gamma((3,5)\to(2,5)) & \gamma((1,4)\to(2,5)) & 1 & \gamma((0,0)\to(2,5)) \\ \gamma((1,2)\to(5,6)) & \gamma((3,4)\to(5,6)) & \gamma((3,5)\to(5,6)) & \gamma((1,4)\to(5,6)) & \gamma((3,6)\to(5,6)) & \gamma((0,1)\to(5,6)) \end{pmatrix}$$

A_5 example cont.

The number of regions in C_w is equal to

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \binom{0}{0}-\binom{0}{2} \\ \binom{2}{1}-\binom{2}{3} & 1 & 0 & 0 & 0 & \binom{4}{2}-\binom{4}{4} \\ \binom{3}{2}-\binom{3}{4} & 0 & 1 & \binom{1}{0}-\binom{1}{4} & 0 & \binom{5}{3}-\binom{5}{5} \\ 0 & 0 & 0 & 1 & 0 & \binom{2}{2}-\binom{2}{4} \\ \binom{4}{3}-\binom{4}{5} & 0 & 0 & \binom{2}{1}-\binom{2}{5} & 1 & \binom{6}{4}-\binom{6}{6} \\ \binom{8}{4}-\binom{8}{6} & \binom{4}{2}-\binom{4}{4} & \binom{3}{1}-\binom{3}{4} & \binom{6}{2}-\binom{6}{6} & \binom{2}{0}-\binom{2}{4} & \binom{9}{4}-\binom{9}{7} \end{pmatrix}$$

A_5 example cont.

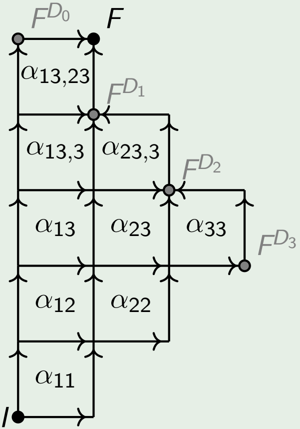
The number of regions in C_w is equal to

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 5 \\ 3 & 0 & 1 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 2 & 1 & 14 \\ 42 & 5 & 3 & 14 & 1 & 132 \end{pmatrix} = 38$$

Type B

$$\gamma\left((a,b) \to F^{D\Sigma}\right) = \begin{cases} \sum_{i=1}^{n} \gamma\left((a,b) \to F^{D_i}\right) & \text{if } b \neq 2n-a+1, 2n-a \\ 1 & \text{if } b = 2n-a+1, 2n-a \end{cases}$$

Example



Type B Determinant

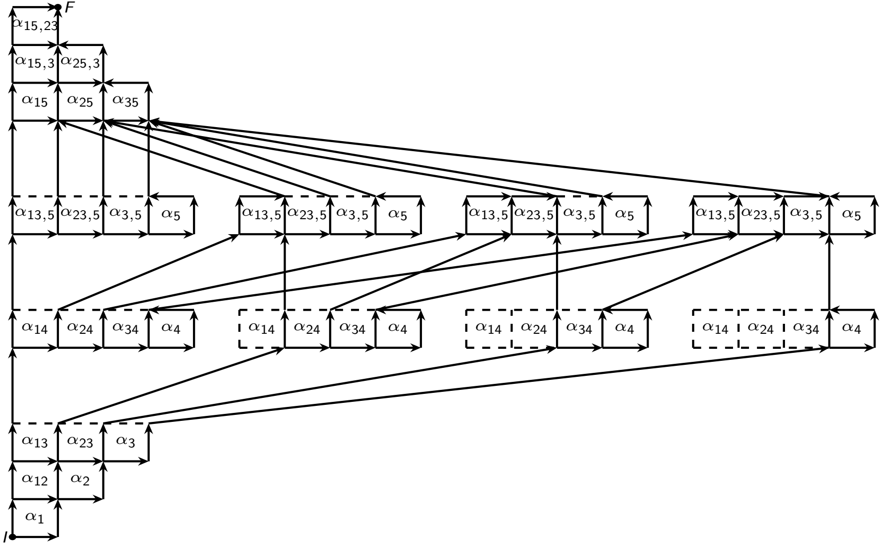
Theorem (D., Tzanaki 2023)

For W type B and inversion set $N(w^{-1}) = \{\alpha_1, \dots, \alpha_k\}$ for $w \in W$. The number of regions in C_w is:

$$\det \begin{pmatrix} 1 & \gamma\left(v_2^{tr} \to v_1^{bl}\right) & \cdots & \gamma\left(v_k^{tr} \to v_1^{bl}\right) & \gamma\left(I \to v_1^{bl}\right) \\ \gamma\left(v_1^{tr} \to v_2^{bl}\right) & 1 & \cdots & \gamma\left(v_k^{tr} \to v_2^{bl}\right) & \gamma\left(I \to v_2^{bl}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma\left(v_1^{tr} \to v_k^{bl}\right) & \gamma\left(v_2^{tr} \to v_k^{bl}\right) & \cdots & 1 & \gamma\left(I \to v_k^{bl}\right) \\ \gamma\left(v_1^{tr} \to F^{D\Sigma}\right) & \gamma\left(v_2^{tr} \to F^{D\Sigma}\right) & \cdots & \gamma\left(v_k^{tr} \to F^{D\Sigma}\right) & \gamma\left(I \to F^{D\Sigma}\right) \end{pmatrix},$$

where $I = (0, 1)$, $F = (1, 2n)$, $v_\ell^{tr} = (i_\ell, j_\ell + 1)$ and $v_\ell^{bl} = (i_\ell - 1, j_\ell)$.

Type D


F
$\alpha_{15,23}$
$\alpha_{15,3}$
$\alpha_{25,3}$
α_{15}
α_{25}
α_{35}
$\alpha_{13,5}$
$\alpha_{23,5}$
$\alpha_{3,5}$
α_5
α_{14}
α_{24}
α_{34}
α_4
α_{13}
α_{23}
α_3
α_{12}
α_2
α_1
I

Paths in type D

Let $V_1 = (x_1, y_1)_{(u_1,v_1)}$ and $V_2 = (x_2, y_2)_{(u_2,v_2)}$ be vertices in Γ_{D_n}

$$\gamma(V_1 \to V_2) = \begin{cases} \gamma\left((x_1,y_1)\to(x_2,y_2)\right) & \text{if } u_1 = 1,\ u_2 = 1 \\ \gamma\left((x_1,y_1)\to(v_2-1,n-2)\right) & \text{if } u_1 = 1,\ u_2 = 2 \\ \displaystyle\sum_{i=0}^{\min(x_2,v_2-1)} \gamma\left((x_1,y_1)\to(i,n-2)\right) & \text{if } u_1 = 1,\ u_2 = 3,\ v_2 \neq n-1 \\ \displaystyle\sum_{i=0}^{\min(x_2,v_2-1)} 2\gamma\left((x_1,y_1)\to(i,n-2)\right) & \text{if } u_1 = 1,\ u_2 = 3,\ v_2 = n-1 \\ \gamma_D(V_1 \to V_2) & \text{if } u_1 = 1,\ u_2 = 4 \\ 1 & \text{if } u_1 = 2,\ u_2 = 3,\ x_2 \geq v_1 - 1 \text{ and} \\ & \qquad v_2 = \min(x_1+1, n-1) \\ (n - v_1 + 1)\gamma_B((n-2,n-2) \to (x_2,y_2)) & \text{if } u_1 = 2,\ u_2 = 4,\ x_1 \geq n-2 \\ (x_1 - v_1 + 2)\gamma_B((x_1,n-2) \to (x_2,y_2)) & \\ \quad +2\gamma_B((n-2,n-2) \to (x_2,y_2)) & \\ \quad + \displaystyle\sum_{j=x_1+1}^{n-3} \gamma_B((j,n-2) \to (x_2,y_2)) & \text{if } u_1 = 2,\ u_2 = 4,\ x_1 < n-2 \\ \gamma_B((v_1-1,n-2) \to (x_2,y_2)) & \text{if } u_1 = 3,\ u_2 = 4,\ x_1 < v_1 \\ \gamma_B((x_1,n-2) \to (x_2,y_2)) & \text{if } u_1 = 3,\ u_2 = 4,\ v_1 \leq x_1 < n-1 \\ \gamma_B((n-2,n-2) \to (x_2,y_2)) & \text{if } u_1 = 3,\ u_2 = 4,\ x_1 = n-1 \\ \gamma_B\left((x_1,y_1) \to (x_2,y_2)\right) & \text{if } u_1 = 4,\ u_2 = 4 \\ 0 & \text{otherwise} \end{cases}$$

Where...

$$\gamma_D(V_1 \to V_2) = \sum_{i=0}^{n-2} \sum_{j=1}^{i+1} \sum_{k=0}^{j-1} \mu_j \cdot \lambda_{i,j,k} \cdot \gamma\left((x_1, y_1) \to (k, n-2)\right) \cdot \gamma_B((i, n-2) \to (x_2, y_2))$$

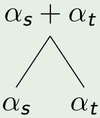
$$\lambda_{i,j,k} = \begin{cases} 2 & \text{if } i = n-2 \\ 1 & \text{if } n-2 > i > j-1 \\ j-k+1 & \text{if } i = j-1 \end{cases}$$

$$\mu_j = \begin{cases} 1 & \text{if } j < n-1 \\ 2 & \text{if } j = n-1 \end{cases}$$

Narayana numbers

The *Narayana number* $N_{P,k}$ is the number of antichains in poset P with cardinality k.

Example



There are 5 antichains:

$$\varnothing, \{\alpha_s\}, \{\alpha_t\}, \{\alpha_s + \alpha_t\}, \{\alpha_s, \alpha_t\}$$

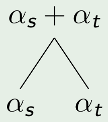
$$N_{(\Phi^+,\leq),0} = 1 \qquad N_{(\Phi^+,\leq),1} = 3 \qquad N_{(\Phi^+,\leq),2} = 1$$

Narayana polynomials

The *Narayana polynomial $N_P(t)$* is the polynomial which keeps track of antichain cardinality:

$$N_P(t) = \sum N_{P,k} t^k$$

Example



$$N_{(\Phi^+,\leq),0} = 1 \qquad N_{(\Phi^+,\leq),1} = 3 \qquad N_{(\Phi^+,\leq),2} = 1$$

$$N_{(\Phi^+,\leq)}(t) = 1 + 3t + t^2$$

Narayana

What if we just applied weights to our corners so that a corner has value t?

$$\gamma(v_1 \to v_2 \to v_3) = t$$

Narayana

What if we just applied weights to our corners so that a corner has value t?

$$\gamma(v_1 \to v_2 \to v_3) = t$$

But,

$$\gamma(v_1 \to v_2 \to v_3) = \gamma(v_1 \to v_2) \cdot \gamma(v_2 \to v_3) = 1 \cdot 1$$

implying $t = 1$ and everything breaks down.

Narayana for Weyl cones

Theorem (D., Tzanaki 2023)

Let Γ_W be digraph associated to W, and Π, I, F as before. Then the Narayana polynomial for a Wel cone is given by:

$$\det \begin{pmatrix} 1 & t{\cdot}\gamma'(F_2\to I_1) & \cdots & t{\cdot}\gamma'(F_n\to I_1) & \gamma'(I\to I_1) \\ t{\cdot}\gamma'(F_1\to I_2) & 1 & \cdots & t{\cdot}\gamma'(F_n\to I_2) & \gamma'(I\to I_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t{\cdot}\gamma'(F_1\to I_n) & t{\cdot}\gamma'(F_2\to I_n) & \cdots & 1 & \gamma'(I\to I_n) \\ t{\cdot}\gamma'(F_1\to F) & t{\cdot}\gamma'(F_2\to F) & \cdots & t{\cdot}\gamma'(F_n\to F) & \gamma'(I\to F) \end{pmatrix}$$

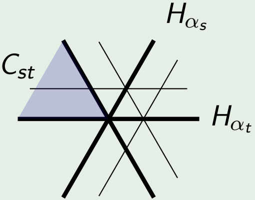
where

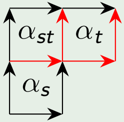
$$\gamma'(v_1 \to v_2) = \sum_{\{\pi \mid I(\pi)=v_1,\, F(\pi)=v_2\}} t^{c(\pi)}, \qquad c(\pi) = \#\ \textit{corners}$$

Narayana polynomial - A_2 example

Example

$w = st,\ N(ts) = \{\alpha_t, \alpha_s + \alpha_t\}$



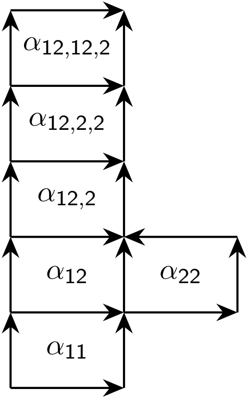


$$\det\begin{pmatrix} 1 & t\gamma'((2,3)\to(0,2)) & \gamma'((0,1)\to(0,2)) \\ t\gamma'((1,3)\to(1,2)) & 1 & \gamma'((0,1)\to(1,2)) \\ t\gamma'((1,3)\to(2,3)) & t\gamma'((2,3)\to(2,3)) & \gamma'((0,1)\to(2,3)) \end{pmatrix}$$

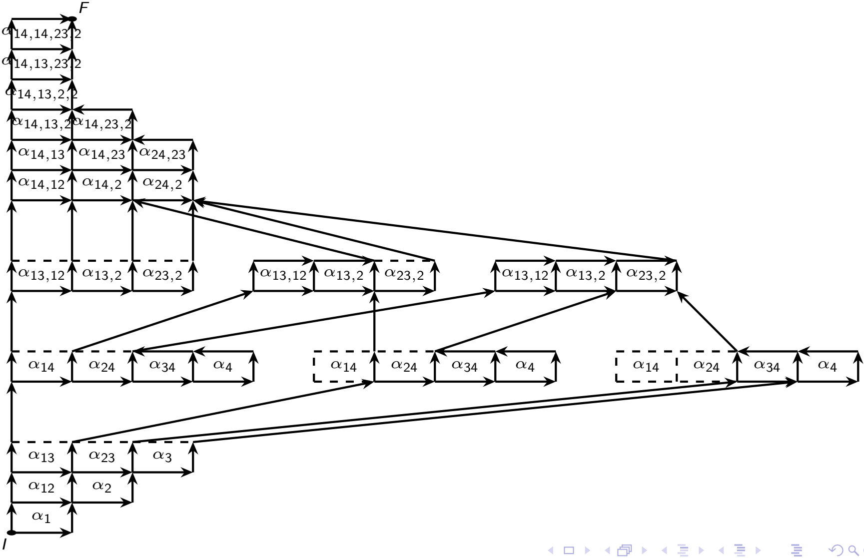
$$= \det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1+t \\ t & t & 1+3t+t^2 \end{pmatrix} = 1 + t$$

THANK
YOU

Digraph - Type G_2



Digraph - Type F_4



Digraphs - Type E_6