

Enumerating Weyl Cones of Shi Arrangements

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Hyperplane - Notation

For $\alpha \in \mathbb{R}^n$ a vector.

- $H_{\alpha,k} = \{v \in \mathbb{R}^n \mid \langle \alpha, v \rangle = k\}$ - hyperplane.
- $H_\alpha = H_{\alpha,0}$ - central hyperplane.
- s_α - reflection fixing H_α pointwise.

Root Systems

Definition

A *root system* Φ is (finite) collection of nonzero vectors satisfying:

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for every $\alpha \in \Phi$.
2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
3. $\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The $\alpha \in \Phi$ are called *roots*.

- Φ^+ - Positive roots
- Φ^- - Negative roots
- Δ - Simple roots
- $W = \langle S \rangle$, $S = \{s_\alpha \mid \alpha \in \Phi^+\}$ - Weyl group.

Coxeter and Shi Arrangements

Definitions

A *Coxeter arrangement* is the arrangement for a root system Φ :

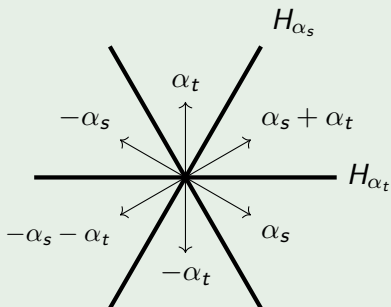
$$\mathcal{A}(\Phi) = \{H_\alpha \mid \alpha \in \Phi^+\}.$$

A *Shi arrangement* is the Coxeter arrangement together with a positive unit translate of each hyperplane:

$$\text{Shi}(\Phi) = \{H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \{0, 1\}\}$$

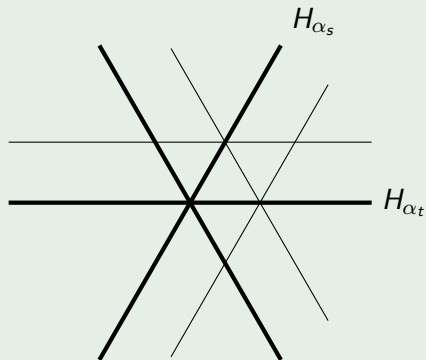
A_2 example

Example (Coxeter Arrangement)



A_2 example

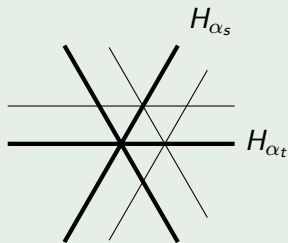
Example (Shi Arrangement)



Regions

A *region* is a (open) connected component of the vector space with the hyperplanes removed.

Example (Shi Arrangement)

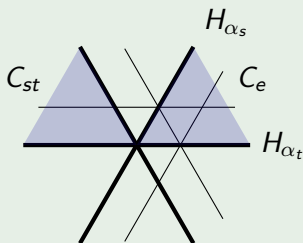


Weyl cone

A *cone* is an intersection of (open) half-spaces of (some) hyperplanes.

For $\text{Shi}(\Phi)$, the regions of the Coxeter subarrangement are in bijection with the elements of W . These regions define cones called *Weyl cones*. The cone associated to the identity is commonly referred to as the *dominant cone*.

Example (Shi Arrangement)



Question:

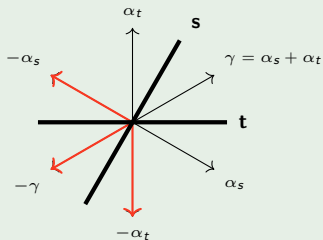
How many regions are in each Weyl cone?

Inversion Sets

The (*left*) *inversion sets* is the set

$$N(w) = \Phi^+ \cap w(\Phi^-).$$

Example



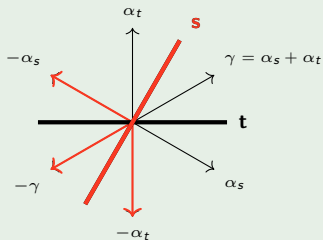
$$\begin{aligned} N(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\} \end{aligned}$$

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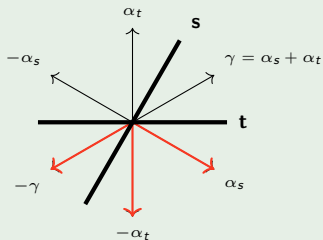
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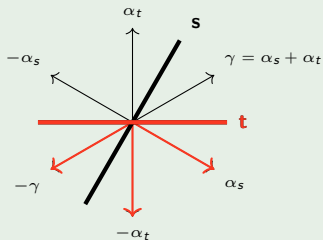
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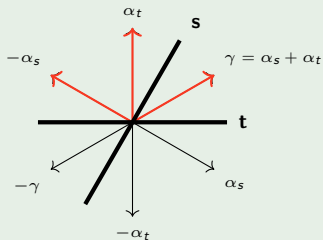
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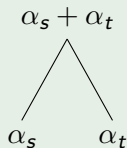
Root Poset

Definition

The *root poset* (Φ^+, \leq) is the poset where

$$\alpha < \beta \iff \beta - \alpha \in \mathbb{N}\Delta$$

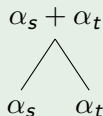
Example



Antichain

An *antichain* in a poset is a set of pairwise incomparable elements.

Example



There are 5 antichains:

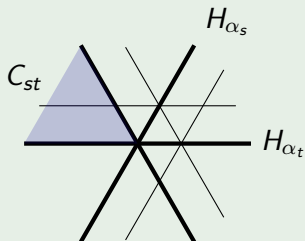
$$\emptyset, \{\alpha_s\}, \{\alpha_t\}, \{\alpha_s + \alpha_t\}, \{\alpha_s, \alpha_t\}$$

Number of regions using antichains

Theorem (Dorpalen-Barry, Stump 2022)

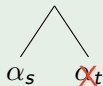
The number of regions in a Weyl cone C_w is equal to the number of antichains in the subposet of the root poset (Φ^+, \leq) restricted to $\Phi^+ \setminus N(w^{-1})$.

Example (A_2 Shi Arrangement)



$$N(ts) = \{\alpha_t, \alpha_s + \alpha_t\}$$

$$\alpha_s * \alpha_t$$



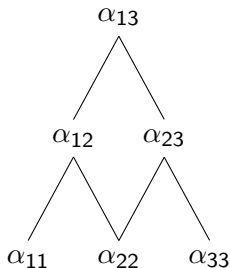
2 antichains: $\emptyset, \{\alpha_s\}$

Diagrams (type A)

Shorthand: $\alpha_{ij} = \sum_{k=i}^j \alpha_k$

α_{13}	α_{23}	α_{33}
α_{12}	α_{22}	
α_{11}		

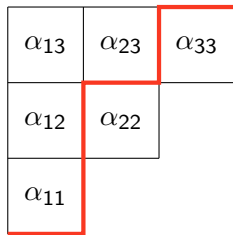
\leftrightarrow



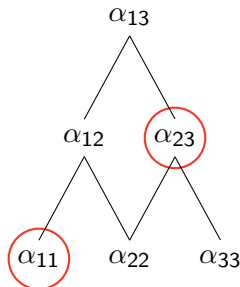
Subdiagrams

A *subdiagram* is a set B of boxes such that if $b \in B$ then every box above and to the left are also in B .

A path in a diagram cuts out a subdiagram associated to an antichain.



\leftrightarrow



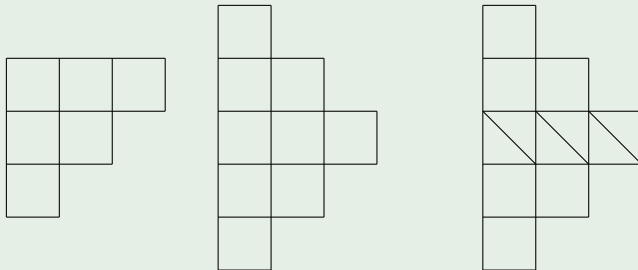
If path touches bottom and right hand edge of box then root is in antichain.

Subdiagrams

Theorem (Shi 1995)

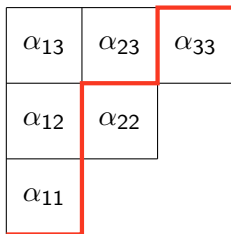
Let Λ be the diagram associated to a Coxeter group W with root system Φ . Then there is a bijection between number of subdiagrams of Λ and antichains in (Φ^+, \leq) .

Example (A_3, B_3, D_4)

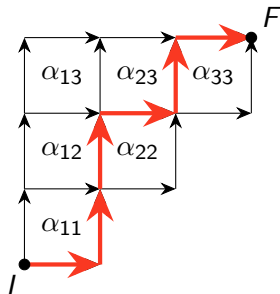


Diagrams to Digraphs - Type A

Shorthand: $\alpha_{ij} = \sum_{k=i}^j \alpha_k$

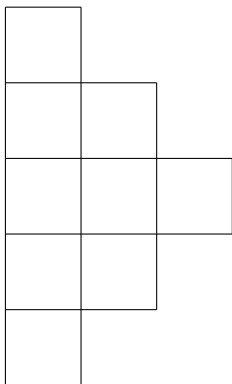


\leftrightarrow

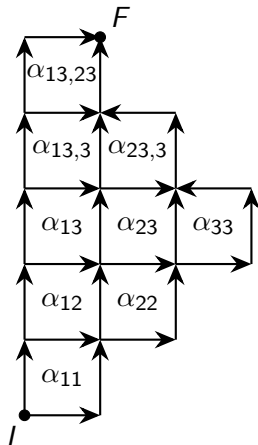


Diagrams to Digraphs - Type B

Shorthand: $\alpha_{ij,kl} = \alpha_{ij} + \alpha_{kl}$



\leftrightarrow

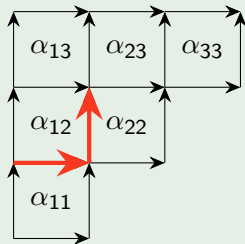


Corners

For each $\alpha \in \Phi^+$ we let Π_α be the set of subpaths of Γ which go under and to the right of α .

Example

$\Pi_{\alpha_{12}}$ is associated to the following subpath.



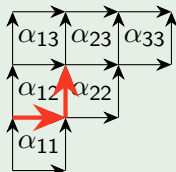
Digraph solution

$$\text{Let } \Pi_w = \bigcup_{\alpha \in N(w^{-1})} \Pi_\alpha$$

Theorem (D., Tzanaki 2023)

Let Γ be the digraph associated to W with root system Φ . There is a bijection between paths in Γ which don't contain subpaths in Π_w and antichains in the root poset (Φ^+, \leq) restricted to $\Phi^+ \setminus N(w^{-1})$.

Example



But..

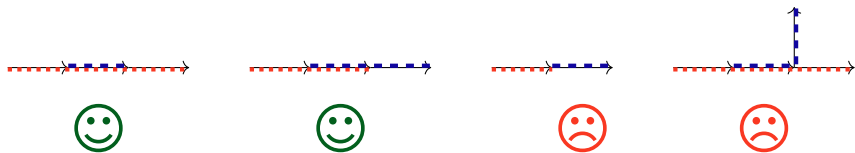
How does this help?

Overlapping paths

Γ a directed graph. $\pi = (v_1, e_1, v_2, e_2, \dots, e_{n-1}, v_n)$ be a path. Let $I_\pi = v_1$ and $F_\pi = v_n$. Γ is *acyclic* if there are no paths such that $I_\pi = F_\pi$.

Two paths π and $\pi' = (u_1, f_1, \dots, f_{m-1}, u_m)$ *overlap* if:

- π is a subpath of π' , or
- there exists some $i \in [n-1]$ such that for all $j \in [n-i]$, then $e_{i+j-1} = f_j$ (the final i edges in π coincide with the first i edges of π').



Number of paths

A collection of paths Π is *non-overlapping* if there does not exist any $\pi, \pi' \in \Pi$ such that π overlaps π' .

Let $\gamma(v \rightarrow v')$ be the number of paths from v to v' .

Theorem (D., Tzanaki 2023)

Let I and F be two arbitrary vertices in an acyclic digraph Γ . Let Π be a collection of non-overlapping paths. Then the number of paths from I to F which do not contain a path in Π as a subpath is equal to:

$$\det \begin{pmatrix} 1 & \gamma(F_2 \rightarrow I_1) & \cdots & \gamma(F_n \rightarrow I_1) & \gamma(I \rightarrow I_1) \\ \gamma(F_1 \rightarrow I_2) & 1 & \cdots & \gamma(F_n \rightarrow I_2) & \gamma(I \rightarrow I_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma(F_1 \rightarrow I_n) & \gamma(F_2 \rightarrow I_n) & \cdots & 1 & \gamma(I \rightarrow I_n) \\ \gamma(F_1 \rightarrow F) & \gamma(F_2 \rightarrow F) & \cdots & \gamma(F_n \rightarrow F) & \gamma(I \rightarrow F) \end{pmatrix}$$

Path enumeration

Theorem (André 1887)

Let Γ be the infinite digraph of \mathbb{Z}^2 with vertical edges pointing north and horizontal edges pointing east. Label every vertex of Γ by its respective coordinates in \mathbb{Z}^2 . Then the number of paths from (x_1, y_1) to (x_2, y_2) weakly above the $x = y$ diagonal is given by:
If $x_1 \leq x_2$ and $y_1 \leq y_2$:

$$\binom{x_2 + y_2 - x_1 - y_1}{y_2 - y_1} - \binom{x_2 + y_2 - x_1 - y_1}{y_2 - x_1 + 1}$$

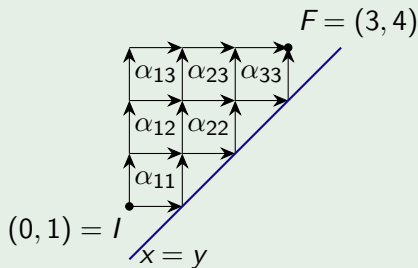
and 0 otherwise.

Type A

Let Γ be the infinite digraph of \mathbb{Z}^2 .

- $I = (0, 1)$ and $F = (n, n + 1)$.
- $\alpha_{ij} = \sum_{k=i}^j \alpha_k \in \Phi, \Rightarrow \pi_{ij} : (i - 1, j) \rightarrow (i, j) \rightarrow (i, j + 1)$.

Example



A_5 example

Let W be the A_5 Coxeter arrangement and $w = s_5 s_2 s_4 s_3 s_1$. Then

$$N(w^{-1}) = \{\alpha_{11}, \alpha_{33}, \alpha_{34}, \alpha_{13}, \alpha_{35}\}$$

$$\alpha_{11} \leftrightarrow (0, 1) \rightarrow (1, 1) \rightarrow (1, 2)$$

$$\alpha_{33} \leftrightarrow (2, 3) \rightarrow (3, 3) \rightarrow (3, 4)$$

$$\alpha_{34} = \alpha_3 + \alpha_4 \leftrightarrow (2, 4) \rightarrow (3, 4) \rightarrow (3, 5)$$

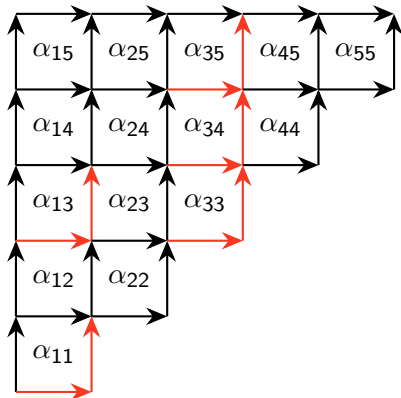
$$\alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3 \leftrightarrow (0, 3) \rightarrow (1, 3) \rightarrow (1, 4)$$

$$\alpha_{35} = \alpha_3 + \alpha_4 + \alpha_5 \leftrightarrow (2, 5) \rightarrow (3, 5) \rightarrow (3, 6)$$

A_5 example cont.

$$N(w^{-1}) = \{\alpha_{11}, \alpha_{33}, \alpha_{34}, \alpha_{13}, \alpha_{35}\}$$

$$\Pi_w = \{\pi_{ij} \mid \alpha_{ij} \in N(w^{-1})\}$$



A_5 example cont.

The number of regions in C_w is equal to

$$\det \begin{pmatrix} 1 & \gamma((3,4) \rightarrow (0,1)) & \gamma((3,5) \rightarrow (0,1)) & \gamma((1,4) \rightarrow (0,1)) & \gamma((3,6) \rightarrow (0,1)) & \gamma((0,1) \rightarrow (0,1)) \\ \gamma((1,2) \rightarrow (2,3)) & 1 & \gamma((3,5) \rightarrow (2,3)) & \gamma((1,4) \rightarrow (2,3)) & \gamma((3,6) \rightarrow (2,3)) & \gamma((0,1) \rightarrow (2,3)) \\ \gamma((1,2) \rightarrow (2,4)) & \gamma((3,4) \rightarrow (2,4)) & 1 & \gamma((1,4) \rightarrow (2,4)) & \gamma((3,6) \rightarrow (2,4)) & \gamma((0,1) \rightarrow (2,4)) \\ \gamma((1,2) \rightarrow (0,3)) & \gamma((3,4) \rightarrow (0,3)) & \gamma((3,5) \rightarrow (0,3)) & 1 & \gamma((3,6) \rightarrow (0,3)) & \gamma((0,1) \rightarrow (0,3)) \\ \gamma((1,2) \rightarrow (2,5)) & \gamma((3,4) \rightarrow (2,5)) & \gamma((3,5) \rightarrow (2,5)) & \gamma((1,4) \rightarrow (2,5)) & 1 & \gamma((0,0) \rightarrow (2,5)) \\ \gamma((1,2) \rightarrow (5,6)) & \gamma((3,4) \rightarrow (5,6)) & \gamma((3,5) \rightarrow (5,6)) & \gamma((1,4) \rightarrow (5,6)) & \gamma((3,6) \rightarrow (5,6)) & \gamma((0,1) \rightarrow (5,6)) \end{pmatrix}$$

A_5 example cont.

The number of regions in C_w is equal to

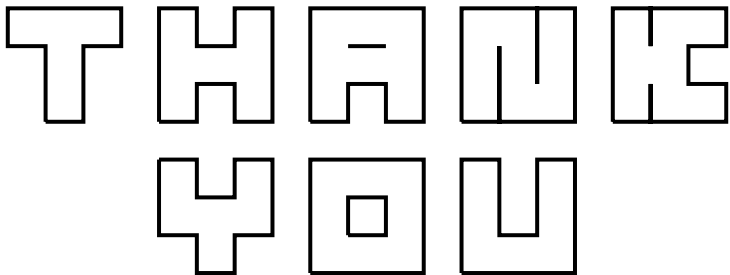
$$\det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \binom{0}{0} - \binom{0}{2} \\ \binom{2}{1} - \binom{2}{3} & 1 & 0 & 0 & 0 & \binom{4}{2} - \binom{4}{4} \\ \binom{3}{2} - \binom{3}{4} & 0 & 1 & \binom{1}{0} - \binom{1}{4} & 0 & \binom{5}{3} - \binom{5}{5} \\ 0 & 0 & 0 & 1 & 0 & \binom{2}{2} - \binom{2}{4} \\ \binom{4}{3} - \binom{4}{5} & 0 & 0 & \binom{2}{1} - \binom{2}{5} & 1 & \binom{6}{4} - \binom{6}{6} \\ \binom{8}{4} - \binom{8}{6} & \binom{4}{2} - \binom{4}{4} & \binom{3}{1} - \binom{3}{4} & \binom{6}{2} - \binom{6}{6} & \binom{2}{0} - \binom{2}{4} & \binom{9}{4} - \binom{9}{7} \end{pmatrix}$$

A_5 example cont.

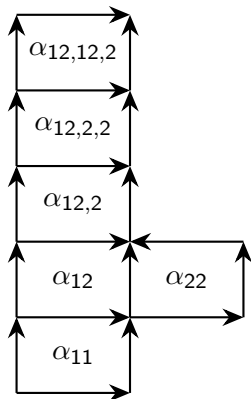
The number of regions in C_w is equal to

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 5 \\ 3 & 0 & 1 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 2 & 1 & 14 \\ 42 & 5 & 3 & 14 & 1 & 132 \end{pmatrix} = 38$$

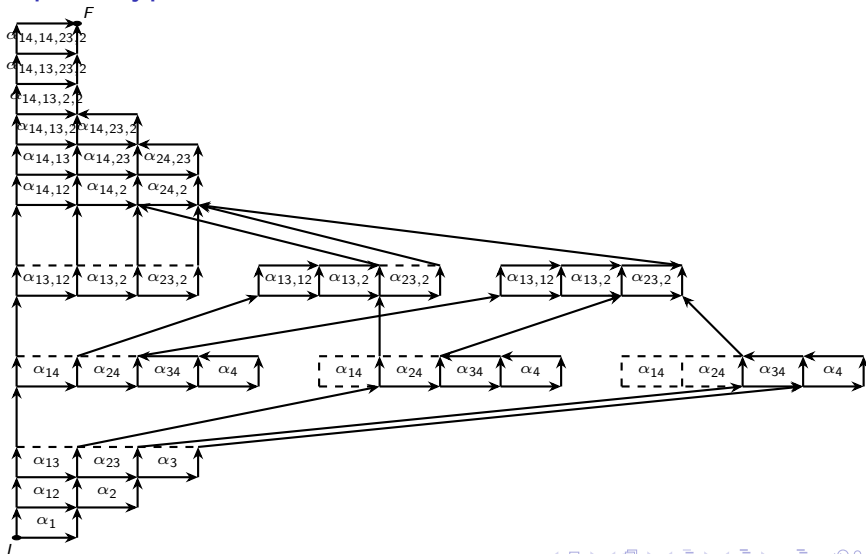
Enumerating Weyl Cones of Shi Arrangements



Digraph - Type G_2



Digraph - Type F_4



Digraphs - Type D_5

