# Maximal degree subposets of $\nu$-Tamari lattices

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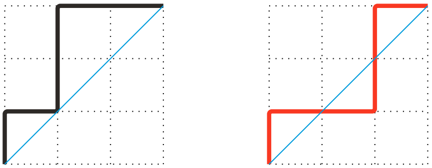
A (brief) history of the main object.

# Dyck paths

*Dyck path of height n:* Path from $(0, 0)$ to $(n, n)$ using only north/east steps, staying weakly above the main diagonal.

Example

Dyck path of height 3:



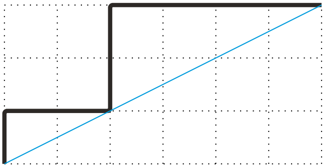
*NENNEE* *NEENNE*

# $m$-Dyck paths

*m-Dyck path of height n (Bergeron and Préville-Ratelle):* Path from $(0, 0)$ to $(n, mn)$ using only north/east steps, staying weakly above the $\frac{1}{m}$ diagonal.

Example

2-Dyck path of height 3:



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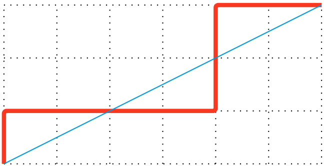


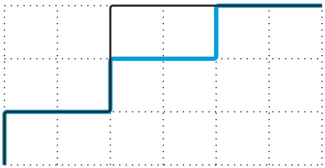
*NEEEENNEE*

## $\nu$-Dyck paths

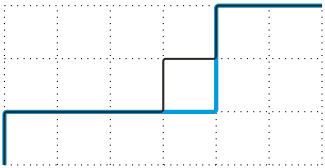
Let $\nu$ be a path from $(0,0)$ to $(s_E, s_N)$ where $s_E, s_N \in \mathbb{N}$.
*$\nu$-Dyck path (Préville-Ratelle and Viennot):* Path from $(0,0)$ to $(s_E, s_n)$ using only north/east steps, staying weakly above $\nu$.

### Example

$\nu$-Dyck paths:



*NEENNEEEE*

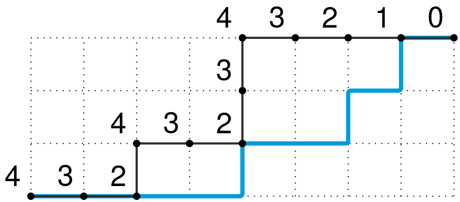


*NEEENENEE*

Dyck paths of height $n$: $\nu = (NE)^n$
$m$-Dyck paths of height $n$: $\nu = (NE^m)^n$

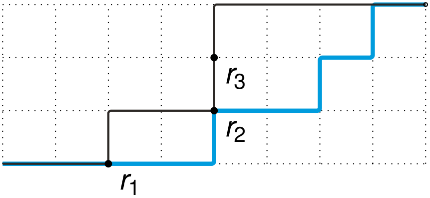
# Horizontal distance

*Horizontal distance*: max number of east steps before passing $\nu$.



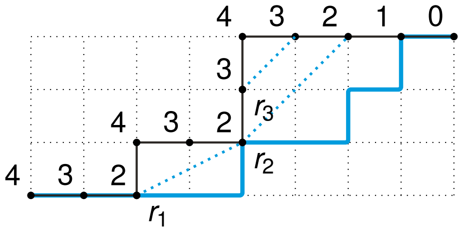
# Right hand points

*$i$-th right hand point $r_i$*: point before $i$-th north step.



# Touch points

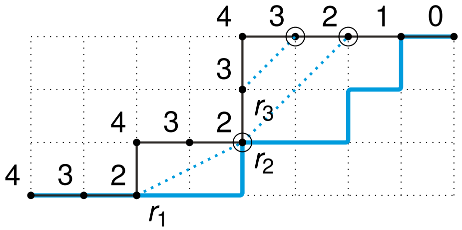
*$i$-th touch point $t_i$*: first point after $r_i$ with the same horizontal distance.




$t_1 = (4, 1)$ $\quad t_2 = (6, 3)$ $\quad t_3 = (5, 3)$

## Touch points

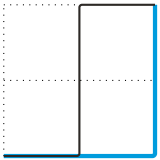
*i-th touch point* $t_i$: first point after $r_i$ with the same horizontal distance.




$t_1 = (4, 1) \quad t_2 = (6, 3) \quad t_3 = (5, 3)$

# Touch points - Pop quiz

Where is $t_1$?



## Touch points - Pop quiz

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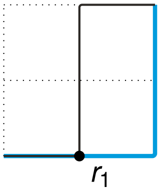

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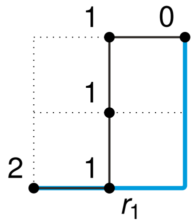
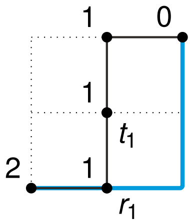


## Touch points - Pop quiz

Where is $t_1$?



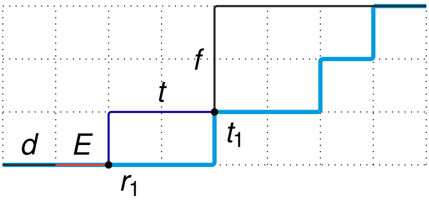

$$t_1 = (1, 1)$$

## From one path to another

Let $D$ be a $\nu$-Dyck path.
If $r_i$ is preceded by an east step, then let:

$$d = \text{subword of } D \text{ from } (0,0) \text{ up until the east step before } r_i$$
$$t = \text{subword of } D \text{ from } r_i \text{ up until } t_i$$
$$f = \text{subword of } D \text{ from } t_i \text{ up until } (s_E, s_N)$$

In other words: $D = dEtf$



# From one path to another

$$
\begin{aligned}
d &= \text{subword of } D \text{ from } (0,0) \text{ up until the east step before } r_i \\
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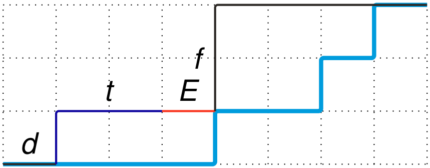
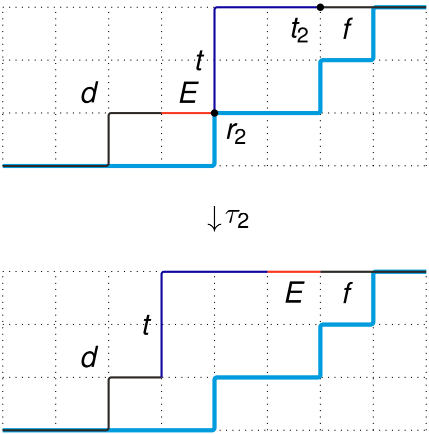
f &= \text{subword of } D \text{ from } t_i \text{ up until } (s_E, s_N) \\
D &= dEtf
\end{aligned}
$$

Putting the $E$ before $r_i$ after $t_i$ we get a new $\nu$-Dyck path: $\tau_i(D) = dtEf$.

$\tau_1(D) =$ 

## From one path to another



## $\nu$-Tamari order

Let $\mathbb{T}_\nu$ be the poset whose elements are $\nu$-Dyck paths, ordered by:

$$D \lessdot_T \tau_i(D) \text{ whenever } \tau_i(D) \text{ exists.}$$

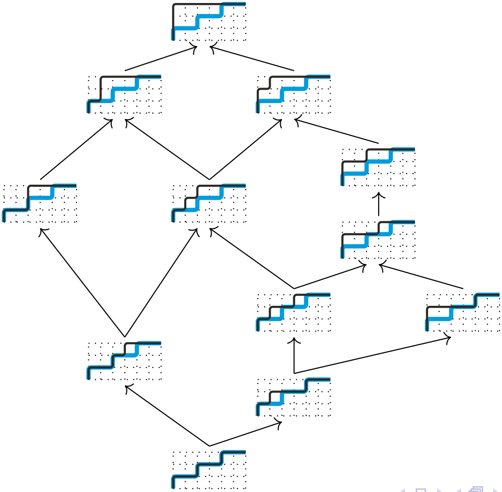
**Theorem (Préville-Ratelle, Viennot 2014)**

*For any path $\nu$, the poset $\mathbb{T}_\nu$ is a lattice.*

Therefore, $\mathbb{T}_\nu$ is called the *$\nu$-Tamari lattice*.

## $\nu$-Tamari lattice example

Let $\nu = NEENEENEE$. (These are 2-Dyck paths of height 3.)



## Maximal degree subposet

Let $D$ be a $\nu$-Dyck path.

*out-degree of D*: number of elements which cover $D$.

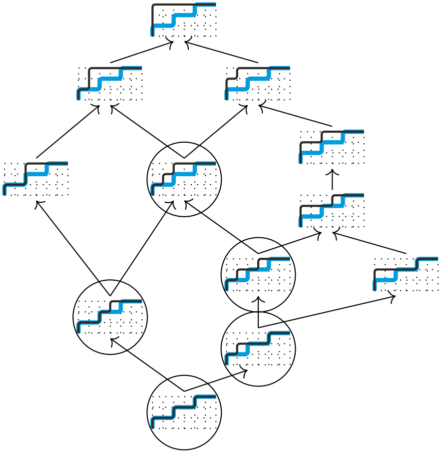
*in-degree of D*: number of elements which are covered by $D$.

$\mathbb{T}_{\nu_{out}}$ = subposet of $\mathbb{T}_\nu$ which have maximal out-degree.

$\mathbb{T}_{\nu_{in}}$ = subposet of $\mathbb{T}_\nu$ which have maximal in-degree.

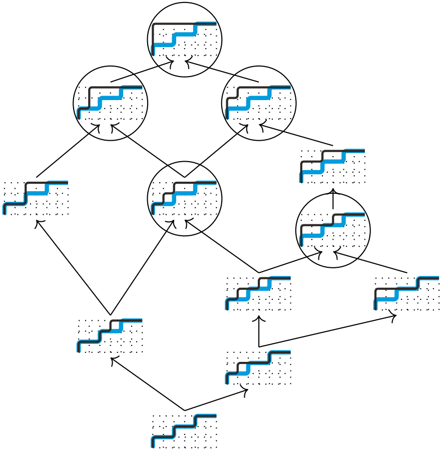
## Out-degree example

Let $\nu = NEENEENEE$. (These are 2-Dyck paths of height 3.)

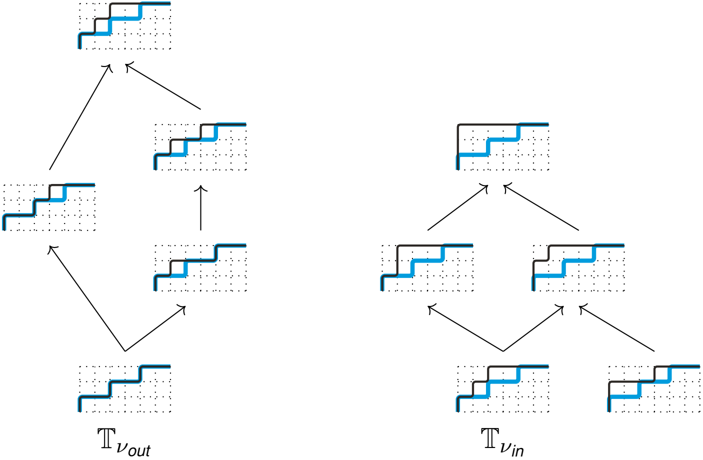


## In-degree example

Let $\nu =$ *NEENEENEE*. (These are 2-Dyck paths of height 3.)

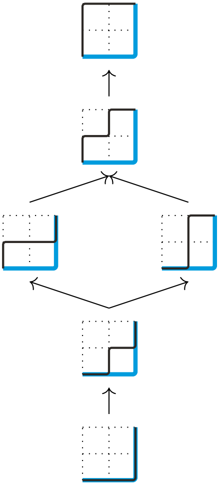


# The maximal subposets



# Another $\nu$-Tamari lattice

Let $\nu = EENN$.

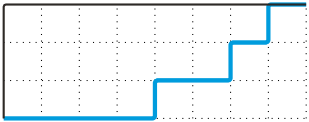


What are the maximal in/out-degrees?

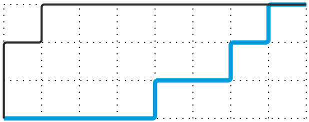
## Staircase shape $\nu$-Dyck path

A $\nu$-Dyck path $D$ is a *staircase shape of size $n$* if $D = N^a(EN)^nE^b$ with $a \geq 0$ and $b \geq 0$.

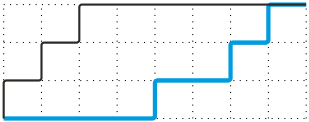
### Example



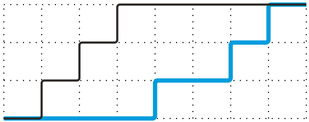
$n = 0$



$n = 1$



$n = 2$



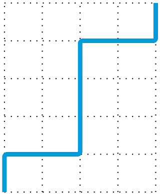
$n = 3$

## Staircase shape algorithm

$\Lambda_i$ = number boxes to the left of $\nu$ in $i$th row. $\Lambda = (\Lambda_1, \ldots, \Lambda_{s_N})$.
Induct on $i$ starting with $i = 1$.

1. Find first $j \in \mathbb{Z}$ such that $i \leq j$.
2. If no such $j$ exists, done! Else, assume $j$ in $d$-th component and replace with $i$.
3. For all other components which are equal to $i$, replace with 0 and pull zeroes forward.
4. Let $\Lambda$ be this new vector and proceed with induction

**Example**



$(0, 2, 2, 2, 4)$

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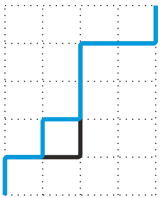

$$(0, 2, 2, 2, 4) \rightarrow (0, 1, 2, 2, 4)$$

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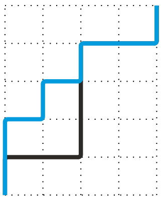
Example



$(0, 2, 2, 2, 4) \to (0, 1, 2, 2, 4)$
$\to (0, 0, 1, 2, 4)$

## Staircase shape algorithm

$\Lambda_i$ = number boxes to the left of $\nu$ in $i$th row. $\Lambda = (\Lambda_1, \ldots, \Lambda_{s_N})$.
Induct on $i$ starting with $i = 1$.

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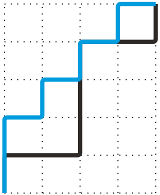
Example



$$(0, 2, 2, 2, 4) \to (0, 1, 2, 2, 4) \to (0, 0, 1, 2, 4) \to (0, 0, 1, 2, 3)$$

## Maximal staircase shape $\nu$-Dyck path

Let $\xi_\nu$ be the staircase shape $\nu$-Dyck path of maximal size $\sigma_\nu$.

**Theorem (D. 2022)**

*Let $\nu$ be a path. The maximal in-degree and maximal out-degree in $\mathbb{T}_\nu$ is equal to $\sigma_\nu$.*

## Maximal staircase shape $\nu$-Dyck path

Let $\xi_\nu$ be the staircase shape $\nu$-Dyck path of maximal size $\sigma_\nu$.

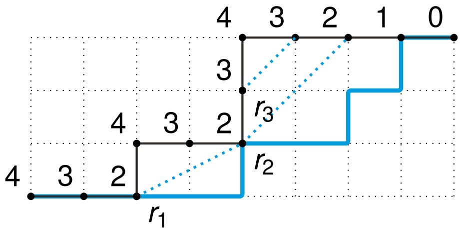
### Theorem (D. 2022)

*Let $\nu$ be a path. The maximal in-degree and maximal out-degree in $\mathbb{T}_\nu$ is equal to $\sigma_\nu$.*

**Proof idea:** Out-degree is relatively easy. Consider the $r_i$ with east steps before.

# Hit points

*$i$-th hit point $h_i$*: first point after $r_i$ with the same horizontal distance *that is followed by an east step or is the final point*.



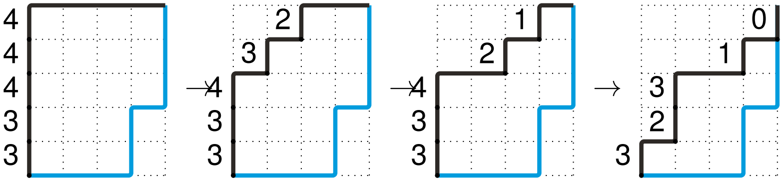

$$h_1 = (6,3) \quad h_2 = (6,3) \quad h_3 = (5,3)$$

## Going down in the $\nu$-Tamari order

**Theorem (D. 2022)**

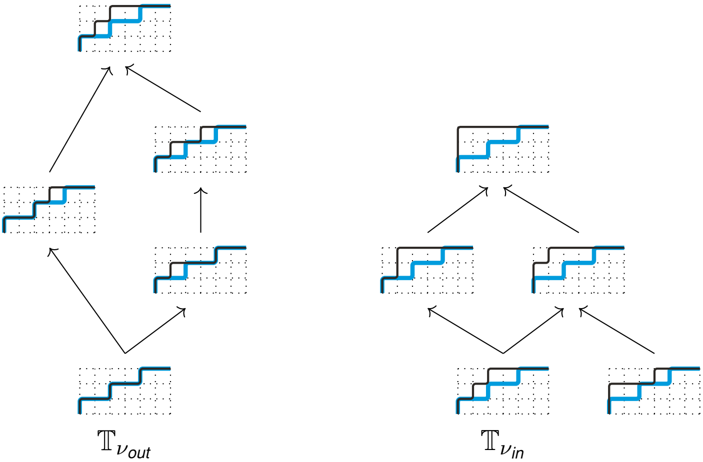
*Let D be a $\nu$-Dyck path and let $r_i$, $t_i$ and $h_i$ be its respective right hand, touch and hit points. Then there exists a $\nu$-Dyck path E such that $E \lessdot_T D$ if and only if there exists i such that $t_i = h_i$ and horizontal distance of $r_i$ is non-zero. In this case $\tau_i(E) = D$.*

## In-degree proof idea



These posets look familiar!

## The maximal subposets



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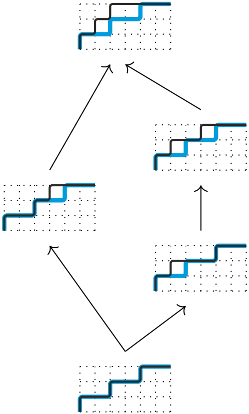
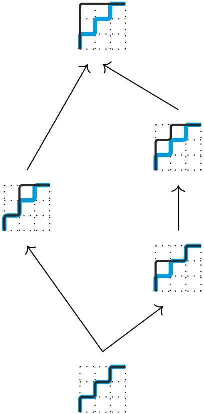


$\nu = (NE^2)^3; \mathbb{T}_{\nu_{out}}$



$\nu = (NE)^3$

## Out-degree subposet

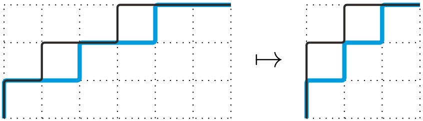
We focus on $m$-Dyck paths of height $n$.
Let $\mathbb{T}_{n,m}$ by the $\nu$-Tamari lattice for $\nu = (NE^m)^n$.

**Theorem (D. 2022)**

*There exists a poset isomorphism:*

$$\mathbb{T}_{n,m_{out}} \to \mathbb{T}_{n,m-1}$$

The map "removes"/"adds" the maximal staircase in the top left corner.



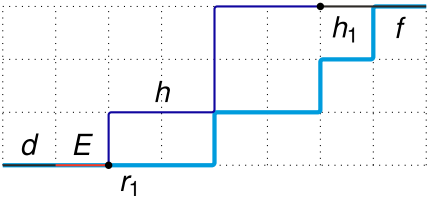
What about in-degree subposet?

## From one path to another

Let $D$ be a $\nu$-Dyck path.
If $r_i$ is preceded by an east step, then let:

$d$ = subword of $D$ from $(0, 0)$ up until the east step before $r_i$
$h$ = subword of $D$ from $r_i$ up until $h_i$
$f$ = subword of $D$ from $h_i$ up until $(s_E, s_N)$

In other words: $D = dEhf$



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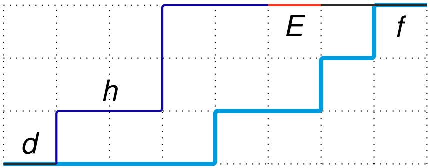
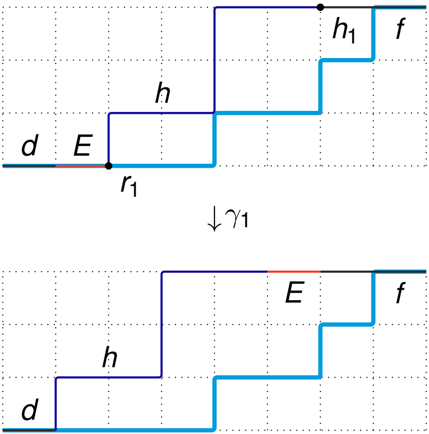
$$D = dEhf$$

Putting the $E$ before $r_i$ after $h_i$ we get a new $\nu$-Dyck path: $\gamma_i(D) = dhEf$.

$\gamma_1(D) =$



## From one path to another



## $\nu$-Greedy order

Let $\mathbb{G}_\nu$ be the poset whose elements are $\nu$-Dyck paths, ordered by:

$$D \lessdot_G \gamma_i(D) \text{ whenever } \gamma_i(D) \text{ exists.}$$

$\mathbb{G}_\nu$ is called the *$\nu$-Greedy poset*.

**Note:** This is the same as the $\nu$-Tamari order except we use hit points instead of touch points.

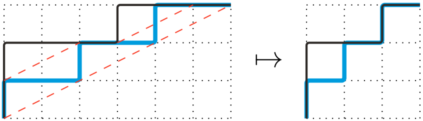
# In-degree subposet

Theorem (D. 2022)

*There exists a poset isomorphism:*

$$\mathbb{T}_{n,m_{in}} \to \mathbb{G}_{n,m-1}$$

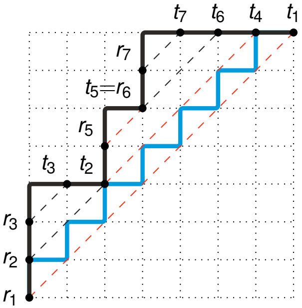
The map removes an east step before each hit point. The reverse map adds an east step at each touch point.



# Touch distance function

## Definition (Bousquet-Mélou, Fusy, Préville-Ratelle 2012)

Let $D$ be a Dyck path. If $a$ and $b$ have same horizontal distance, $\ell(a, b)$ is diagonal distance between them.
*Touch distance vector* $d_D^t = (\ell(r_1, t_1), \ldots, \ell(r_n, t_n))$.




$$d_D^t = (7, 2, 1, 4, 1, 2, 1)$$

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Theorem (Bousquet-Mélou, Fusy, Préville-Ratelle 2012)

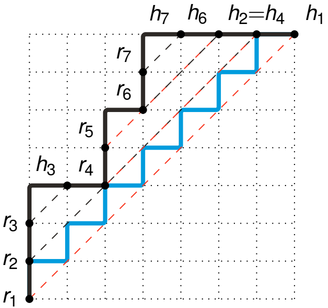
$$d_D^t \leq d_{D'}^t \iff D \leq_T D'$$

# Hit distance function

## Definition

Let $D$ be a Dyck path. If $a$ and $b$ have same horizontal distance, $\ell(a, b)$ is diagonal distance between them.

*Hit distance vector* $d_D^h = (\ell(r_1, h_1), \ldots, \ell(r_n, h_n))$.




$$d_D^h = (7, 6, 1, 4, 3, 2, 1)$$

$$d_D^t = (7, 2, 1, 4, 1, 2, 1)$$

# Hit distance function

**Definition**

Let $D$ be a Dyck path. If $a$ and $b$ have same horizontal distance, $\ell(a, b)$ is diagonal distance between them.
*Hit distance vector* $d_D^h = (\ell(r_1, h_1), \ldots, \ell(r_n, h_n))$.

**Theorem (D. 2022)**

$$D \leq_G D' \iff d_D^h \leq d_{D'}^h \text{ and } d_D^t \leq d_{D'}^t$$

# Proof idea

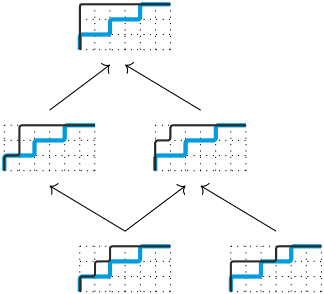
**Theorem (D. 2022)**

*There exists a poset isomorphism:*

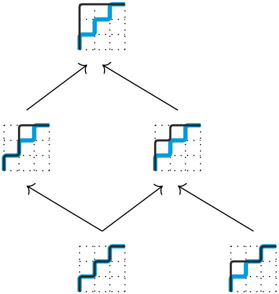
$$\mathbb{T}_{n,m_{in}} \to \mathbb{G}_{n,m-1}$$

- Greedy to Tamari is “easy”.
- Convert from $m$-Dyck path of height $n$ to Dyck path of height $mn$.
- Do conversion to smaller Dyck paths in that world.
- Show Greedy order holds using distance functions.
- Go to the world of $m-1$-Dyck paths of height $n$.

# The maximal subposets



$\mathbb{T}_{3,2_{in}}$



$\mathbb{G}_{3,1}$

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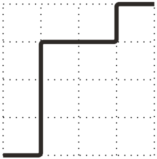
Unfortunately, there are $\nu$ for which $\mathbb{T}_{\nu_{in}}$ is not in *set* bijection with any $\nu'$ weakly above $\nu$.

Example

$\nu = ENNNEENE$



There are 6 $\nu$-Dyck paths with maximal in-degree.

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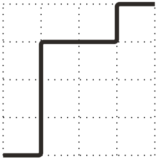

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## Conjecture

*The number of elements in* $\mathbb{T}_{\nu_{in}}$ *is equal to the number of elements in* $\mathbb{T}_{\nu_{out}}$.

Thanks!