Welcome!

Thanks for coming to my poster talk! You can either go through the slides like "normal", or jump around using the links in green (ex: Go to directory) or in the bottom-right corner of every slide .

If you have any questions, don't hesitate to ask Aram!

▸ Start with the directory

▸ Start with the main result!

Link to directory

arXiv: 2008.03794

Directory of contents

Background:

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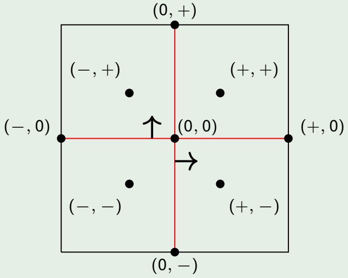
Come back at any time

Sign vectors

- A *sign vector* is a vector in $\mathcal{V}_n = \{+, 0, -\}^n$ which give the sign of a generic point in $\mathbb{R}^n$.

Example

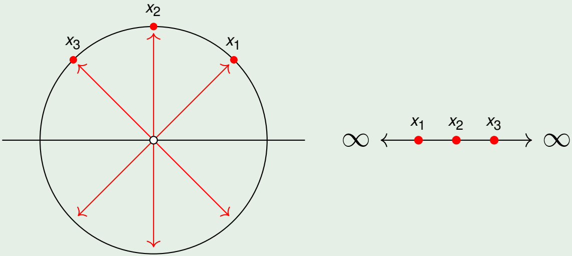


Real Projective Space

- *Real Projective space* $\mathbb{P}^n$ is quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ under equivalence relation $x \sim \lambda x$ for $\lambda \in \mathbb{R}$.

Example

$\mathbb{P}^1$



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Projective Sign Vectors

- A *projective sign vector* is an equivalence class of sign vectors in projective space where $\omega \sim \omega'$ iff $\omega = \omega'$ or $\omega = -\omega'$.

$$\mathcal{PV}_n := \left(\mathcal{V}_n \setminus \{0\}^n\right) / \sim \;\cong \{\omega \in \mathcal{V}_n \;:\; \text{First non-zero entry of } \omega \text{ is } +\}.$$

- Let P_n denote the poset $(\mathcal{PV}_n, <)$ where for $\omega, \omega' \in \mathcal{PV}_n$:

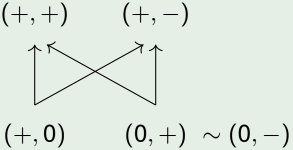
$$\omega' \leq \omega \iff \pm\omega' \subseteq \omega$$

Example

$$\mathcal{V}_2 = \{(+,+), (+,0), (+,-), (0,+), (0,0), (0,-), (-,+), (-,0), (-,-)\}$$

$$\downarrow$$

$$\mathcal{PV}_2 = \{(+,+), (+,0), (+,-), (0,+)\}$$

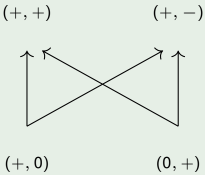


P_2

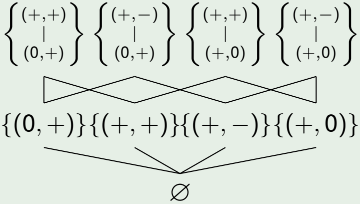
Order complex (of a poset)

- *Simplicial complex* Δ - A collection of sets s.t. $\sigma \in \Delta$ and $\tau \subseteq \sigma$ implies $\tau \in \Delta$.
- The sets are called *faces*. Maximal sets are called *facets*.
- *Order complex* $\Delta(P)$ of a poset P - Simplicial complex where faces are chains in P.

Example



P_2

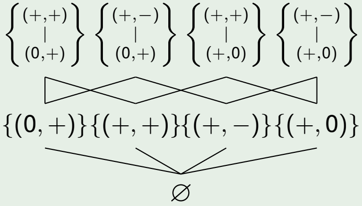


$\Delta(P_2)$

f-vector and h-vector

- Δ a d-dim simplicial complex.
- *f-vector* is the vector $f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_d)$ where f_i = number of i-dim faces.
- *h-vector* is the vector $h(\Delta) = (h_0, h_1, \ldots, h_{d+1})$ where $h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}$.

Example



$\Delta(P_2)$

$$f(\Delta(P_2)) = (1, 4, 4)$$
$$h(\Delta(P_2)) = (1, 2, 1)$$

Partitionable

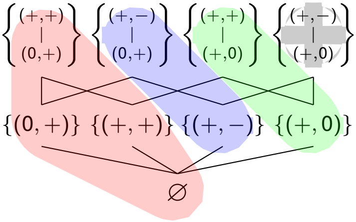
A simplicial complex Δ is *partitionable* if $\Delta = \bigsqcup [G_i, F_i]$ where F_i is a facet.

Proposition (Stanley)

Let Δ be a partitionable simplicial complex. Then

$$h_i(\Delta) = |\{j \ : \ |G_j| = i\}|.$$

$h(\Delta(P_2)) = (1, 2, 1)$



$\Delta(P_2)$

Coxeter groups

Type A_n

The elements in type A_n Coxeter groups can be represented as permutations in $\mathfrak{S}_{n+1}$.

Example

$5 7 2 3 8 1 4 6 \in A_7$

Type B_n

The elements in type B_n Coxeter groups can be represented as *signed* permutations of $\mathfrak{S}_n$.

Example

$5\bar{7}23\bar{8}\bar{1}46 \in B_8$

Type D_n

The elements in type D_n Coxeter groups can be represented as *even signed* permutations of $\mathfrak{S}_n$.

Example

$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$

Descents

Type A_n

For $\pi = \pi_1 \dots \pi_{n+1}$ in A_n let $\mathrm{des}_A(\pi)$ denote the descent set of π.

$$\mathrm{des}_A(\pi) = \{i : \pi_i > \pi_{i+1}\}$$

Example

$\pi = 57238146 \in A_7$

$\mathrm{des}_A(\pi) = \{2, 5\}$

Type B_n

For $\pi = \pi_1 \dots \pi_n$ in B_n let $\mathrm{des}_B(\pi)$ denote the descent set of π.

$$\mathrm{des}_B(\pi) = \mathrm{des}_A(0\pi)$$

Example

$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$

$\mathrm{des}_B(\pi) = \{1, 3, 4\}$

Type D_n

For $\pi = \pi_1 \dots \pi_n$ in D_n let $\mathrm{des}_D(\pi)$ denote the descent set of π.

$$\mathrm{des}_D(\pi) = \mathrm{des}_A(\bar{\pi}_2\pi)$$

Example

$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$

$\mathrm{des}_D(\pi) = \{0, 1, 3, 4\}$

Cyclic Sign Variations

Given a sign vector $\omega \in \mathcal{V}_n = \{+, 0, -\}^n$.

$$\mathrm{cvar}(\omega) = \text{number of times } \omega \text{ changes sign, cyclically}$$

$i \in [n]$ is a *cyclic sign flip* of ω if there exists a j such that $\omega_{i-j}\omega_i < 0$ while $\omega_{i-k}\omega_i = 0$ for all $1 \leq k < j$ where $\omega_i = \omega_{i+n}$.

Example

$$\omega = (+, +, -, -, -, -, +, -) \Rightarrow \mathrm{cvar}(\omega) = 4$$

$$\underset{1}{(+}, \underset{2}{+}, \underset{3}{-}, \underset{4}{-}, \underset{5}{-}, \underset{6}{0}, \underset{7}{+}, \underset{8}{-}) \quad \leftrightarrow \quad \{1, 3, 7, 8\}$$

Main Theorem

Theorem (Bergeron, D., Machacek 2020)

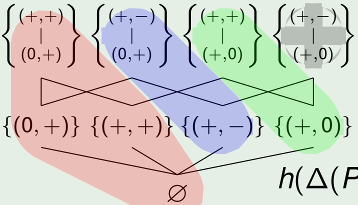
The order complex $\Delta(P_n)$ is partitionable with h-vector

$$h_i(\Delta(P_n)) = |\{\pi \in D_n \ : \ |\mathrm{des}_B(\pi)| = i\}| \quad \textit{for all } 0 \leq i \leq n$$

with Coxeter group of type D_n and descent set, des_B, of type B.

Example ($n = 2$)

$\pi \in D_n$	$\mathrm{des}_B(\pi)$
12	$\varnothing$
$\bar{2}\bar{1}$	$\{0\}$
21	$\{1\}$
$\bar{1}\bar{2}$	$\{0, 1\}$



$$h(\Delta(P_2)) = (1, 2, 1)$$

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Even signed permutations and chains (proof idea)

1. Use cyclic sign variations on the set of negative numbers to get a sign vector.
2. For each set of increasing sequences, replace signs with 0s to get a new sign vector.
3. Proceed inductively to get chain.

$$(+,+,-,-,-,-,+,-)$$

$$(+,+,-,-,0,-,+,-)$$

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \overset{\min}{\longleftrightarrow} (+,0,-,-,0,-,0,-)$$

$$(57238146, \{1,3,7,8\})$$

$$(+,0,0,-,0,-,0,-)$$

~~$\mathrm{des}_D(\pi) = \{0,1,3,4\}$~~

$$\mathrm{des}_B(\pi) = \{1,3,4\}$$

Restriction of variations

- $\mathcal{PV}_{n,m} = \{\omega \in \mathcal{PV}_n \;:\; \mathrm{var}(\omega) \leq m\}$.
- $P_{n,m} = (\mathcal{PV}_{n,m}, <)$.
- $D_{n,m} = \{\pi \in D_n \;:\; \pi$ has at most m negatives$\}$.

Theorem (Bergeron, D., Machacek 2020)

If $m \leq n-1$ is even then the order complex $\Delta(P_{n,m})$ is partitionable. Moreover,

$$h_i(\Delta(P_{n,m})) = |\{\pi \in D_{n,m} \;:\; |\mathrm{des}_B(\pi)| = i\}|$$

for each $0 \leq i \leq n$.