Week 9

2–6 Mar 2020

9.1 Differentials - §3.10

Remember how we had said that $\frac{dy}{dx}$ is not a fraction and we shouldn't think of it as one? We're about to change all of that!

We can think of dy as a very small change, an approximation of a larger Δy. Let's just pretend that $f'(x) = \frac{dy}{dx}$ is a fraction and put like terms on the same side. So we have ____________________. We call dx and y ____________________ .

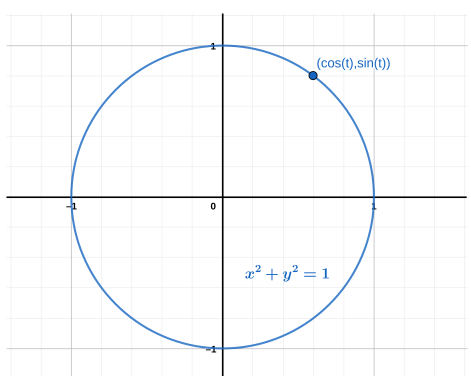
It turns out we can use differentials as a good approximation of Δy. Let's see an example to see what in the world I'm talking about.

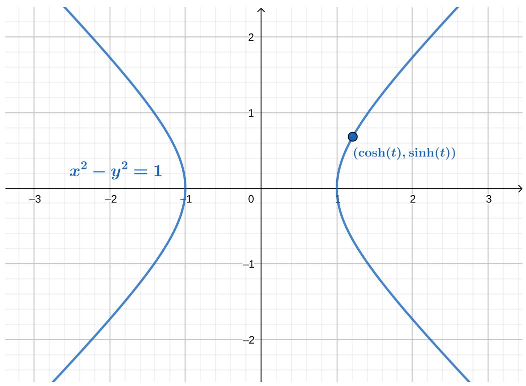
Example 9.1 Let $f(x) = x^2 - 2x + 4$. Let us look at the change from 2 to 2.01.

So this "division" turns out to be a good approximation! Also, finding dy was a lot easier to compute than Δy.

9.2 Hyperbolic functions - §3.11

Let's talk about hyperpbolic functions next. Recall that we can get the sin and cos functions from a circle. If instead we looked at a hyperbola, we would get the hyperbolic functions.

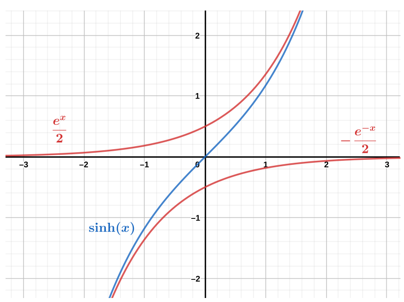


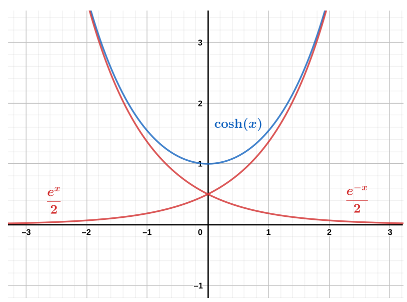


They're hard to remember, so here are their definitions:

$\sinh(x)$	$\frac{e^x - e^{-x}}{2}$	$\operatorname{csch}(x)$	$\frac{1}{\sinh(x)}$
$\cosh(x)$	$\frac{e^x + e^{-x}}{2}$	$\operatorname{sech}(x)$	$\frac{1}{\cosh(x)}$
$\tanh(x)$	$\frac{\sinh(x)}{\cosh(x)}$	$\coth(x)$	$\frac{1}{\tanh(x)}$

Here is what $\sinh(x)$ and $\cosh(x)$ look like





If we're lucky, we'll see some applications of these functions later in the term, but for now let's look at some of the properties of these functions.

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$$
$$\sinh(x+y) = \sinh(x)\cosh(x) + \cosh(x)\sinh(y) \qquad \cosh^2(x) - \sinh^2(x) = 1$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \qquad 1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

We can also compute the derivatives of the hyperbolic functions fairly easily using their original definitions using exponential functions.

Example 9.2 Let $f(x) = \cosh(x)$, what is its derivative?

Here are all the derivatives for the hyperbolic functions.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sinh(x)$	______	$\operatorname{csch}(x)$	______
$\cosh(x)$	______	$\operatorname{sech}(x)$	______
$\tanh(x)$	______	$\coth(x)$	______

Just like the trig functions, hyperbolic functions have inverses too.

$f(x)$	$f^{-1}(x)$	Domain
$\sinh(x)$	$\operatorname{arsinh}(x) = \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$	$\mathbb{R}$
$\cosh(x)$	$\operatorname{arcosh}(x) = \cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$	$[1, \infty)$
$\tanh(x)$	$\operatorname{artanh}(x) = \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$
$\operatorname{csch}(x)$	$\operatorname{arcsch}(x) = \operatorname{csch}^{-1}(x) = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right)$	$\mathbb{R}\setminus\{0\}$
$\operatorname{sech}(x)$	$\operatorname{arsech}(x) = \operatorname{sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0, 1]$
$\coth(x)$	$\operatorname{arcoth}(x) = \coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$

Similarly to the inverse trig functions, the inverse hyperbolic functions have derivatives.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\operatorname{arsinh}(x)$	$\frac{1}{\sqrt{1+x^2}}$	$\operatorname{arcsch}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$\operatorname{arcosh}(x)$	$\frac{1}{\sqrt{x^2-1}}$	$\operatorname{arsech}(x)$	$-\frac{1}{x\sqrt{1-x^2}}$
$\operatorname{artanh}(x)$	$\frac{1}{1-x^2}$	$\operatorname{arcoth}(x)$	$\frac{1}{1-x^2}$

$\operatorname{artanh}(x)$ and $\operatorname{arcoth}(x)$ appear to have the same function as a derivative, but recall that $\operatorname{artanh}(x)$ and $\operatorname{arcoth}(x)$ have different domains, so the derivatives are defined on different parts of the graph.

9.3 Maximum and minimum values - §4.1

A lot of problems in life are optimization problems: when do functions reach maximum/minimum values. We kinda saw this with our astronaut problem where we wanted to find maximum acceleration. We also see this for example when trying to decide:

- What was the peak value of CO_2 in the atmosphere before 0 BCE?
- What time do restaurants have the most amount of clientele?
- What shape of packaging for our product do we need to minimize costs?

There are an endless supply of max/min problems.

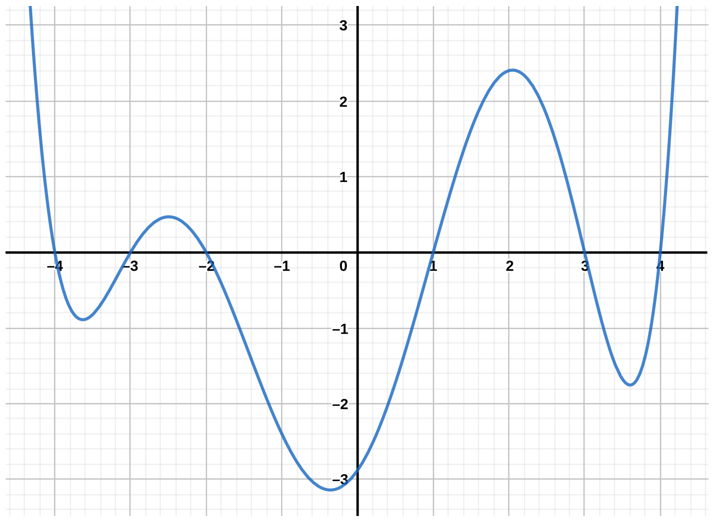
Exercise 9.3 Try and come up with a max/min problem and share it with your neighbor:

Max/Min problem: ______________________________

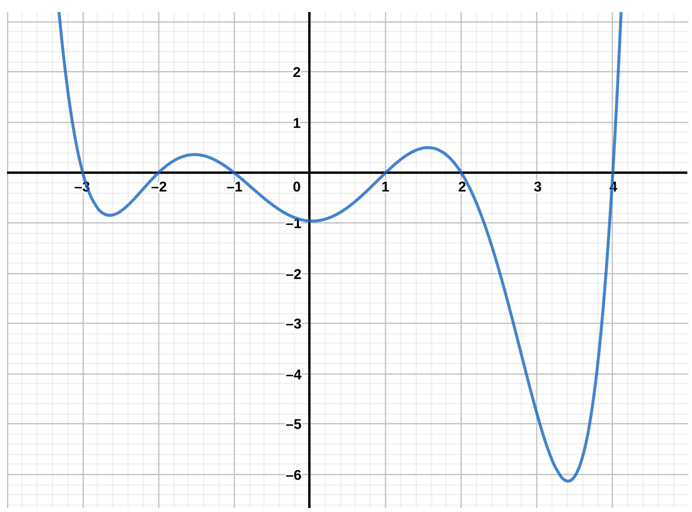
Mathematically, what is maximum and minimum though? A function f has an *absolute maximum value at c on a domain D* if $f(c) \geq f(x)$ for every $x \in D$. Similarly, a function f has an *absolute minimum value at c on a domain d* if $f(c) \leq f(x)$ for every $x \in D$. Absolute values are sometimes known as global values. An *extreme value of f* is an ______________________________ of f.

We can also look at points which are maximal/minimal in a smaller area. If $f(c) \geq f(x)$ for every x near c then we say $f(c)$ is a *local maximum*. Similarly, $f(c)$ is a *local minimum* if $f(c) \leq f(x)$ for every x near c.

Example 9.4



Exercise 9.5 With a partner, fill in the absolute max/min and the local max/mins in the domain $[-3, 4]$ on the following graph:



How do we know that extreme values even exist? Maybe we have some graph where there are no extreme values! This turns out not to be the case as was shown by Bernard Bolzano in the 1830s when he proved the following result.

Theorem 9.6 (Extreme value theorem) *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

What about local maxima and minima? It turns out that Pierre Fermat gave a nice property on local maxima and minima.

Theorem 9.7 (Fermat's Theorem) *If f has a local maximum or minimum at c and if $f'(c)$ exists then $f'(c) = 0$.*

The book (rightfully so) puts up a million warnings about this theorem. It's very easy to want to say that the theorem says more than it actually does. There are 3 common errors that students make with this theorem, two of which are mentioned in the book.

(1) ______________________________

(2) ______________________________

(3) ______________________________

Although there are a lot of easy pitfalls with Fermat's theorem, it does suggest that *maybe*, we can determine how a function looks based on its derivative.

A ____________ *number* of a function f is a number c (in the domain) such that $f'(c) = 0$ or $f'(c)$ does not exist. In other words, Fermat's theorem becomes:

If f has a local maximum or minimum at c then c is a critical number of f.

It turns out we can use cricital numbers to tell us when we have an absolute max or min in some interval $[a, b]$. For this we:

(1) Find the critical values of f in the interval (a, b).

(2) Find the values of f at the critical values and at the end points of the interval (at a and b).

(3) The largest value of the previous step is the absolute maximum and the smallest value is the absolute minimum.

Example 9.8 Let $f(x) = x^3 - x^2 - x$, find its absolute maximum and minimum on the interval $[\frac{-1}{2}, 2]$.

Exercise 9.9 Let $f(x) = x^3 + x^2 - 1$, find its extreme values on the interval $[-1, 1]$.

9.4 The mean value theorem - §4.2

Our next goal is the mean value theorem. We start off with Rolle's theorem, which was named after Michel Rolle who published it in 1691 for polynomial functions. Although Rolle was the first to publish his ideas, Indian mathematician Bhāskara II (1114–1185) is credited to be the first person we know of to have knowledge of the theorem. In addition, the theorem itself was not proved in full until 1823 when Cauchy proved it in its entirety.

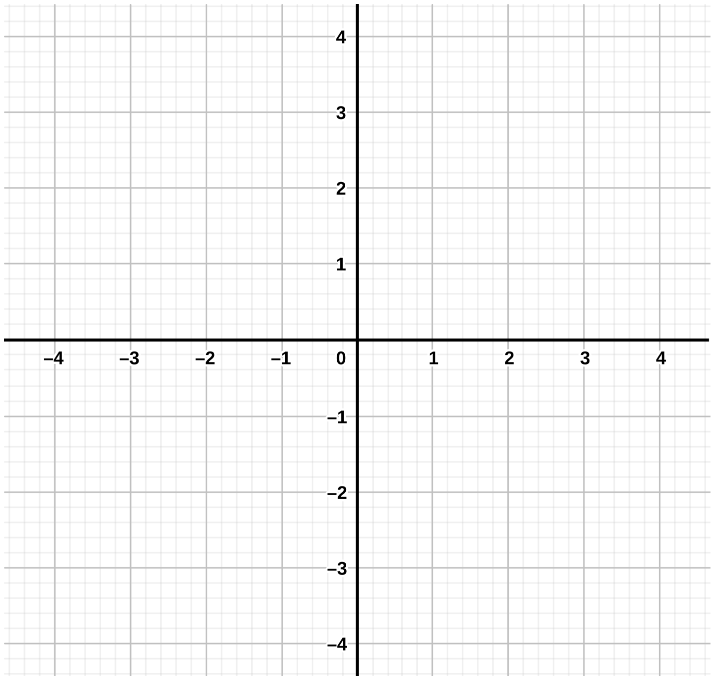
Theorem 9.10 (Rolle's theorem) *Let f be a function that satisfies the following three hypotheses:*

(1) f is continuous on $[a, b]$.

(2) f is differentiable on (a, b).

(3) $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.



Rolle's theorem is used to find the number of roots a function might have.

Example 9.11 Show that the equation $f(x) = x^3$ has exactly one real root.

The mean value theorem was known first to Parameshvara (1370–1460), an Indian mathematician. In Europe, it was first stated by Joseph-Louis Lagrange in the following form.

Theorem 9.12 (Mean value theorem) *Let f be a function such that:*

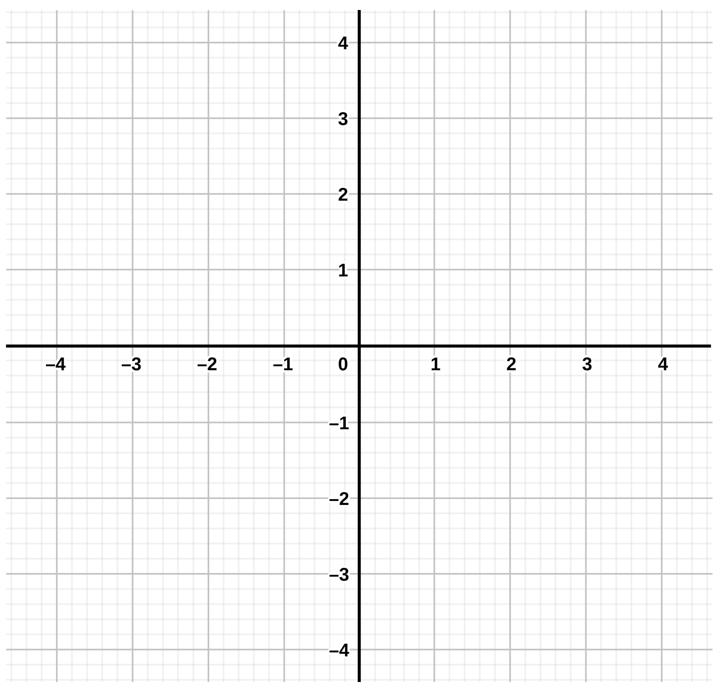
(1) f is continuous on $[a, b]$.

(2) f is differentiable on (a, b).

Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Sometimes this theorem is known as *Lagrange's mean value theorem*. There is an extended version of this theorem which we won't go over called the *extended mean value theorem* or *Cauchy's mean value theorem*.



Example 9.13 Let $f(x) = x^2 - 2$ and let's look at the interval $[0, 2]$. Show the mean value thorem is true on this interval.

The mean value theorem allows us to state the following two theorems.

Theorem 9.14 *If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).*

Corollary 9.15 *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); i.e., $f(x) = g(x) + c$ for some constant c.*