Week 8

24–28 Feb 2020

8.1 Rates of Change - §3.7

In the last couple weeks we've been looking at different ways to apply derivatives into the real world. Most of these came through rates of change.

What is "rate of change"? It means "how much something is changing". This can be interpreted in many different ways! In the astronaut example, we talked about how velocity is the rate of change of position. We talked about acceleration being the rate of change of velocity. In a lot of our other examplese we looked at how much something changes over time and figuring out the instantaneous rate of change at some fixed time. We looked at the following:

- Acceleration/Velocity (Astronauts!)
- Density of objects
- Electricity flow (current and Ohm's law)
- Rates of chemical reactions
- Population growth

We're now going to continue looking at how derivatives help in the real world.

The examples here are the same examples in the book. There are a bunch of even more exciting examples that can be found through a quick google search of applications of derivatives.

Example 8.1 We first look at thermodynamics. In thermodynamics we're looking at how mcuh something can be compressed. So given some amount of pressure, the volume of a substance is going to change. The rate of change is then just the change of volume over pressure: dV/dP. The *isothermal compressibility* is then defined to be:

$$\text{isothermal compressibility } = \beta = -\frac{1}{V}\frac{dV}{dP}$$

In other words, β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases (at constant temperature).

Let $V = \frac{3.5}{P^2}$ and let's figure out the isothermal compressibility when the pressure is 10 kPa (kilopascals). First we must calculate dV/dP.

$$\frac{dV}{dP} = \underline{\hspace{3cm}}$$

Then we calculate β.

$$\beta = -\frac{1}{V}\frac{dV}{dP} = \underline{\hspace{8cm}}$$

So then $\beta(10) = \underline{\hspace{1.5cm}}$.

Example 8.2 Our next example looks at economics. If we let $C(x)$ denote the total cost that a company incurs in producing x number of products then C is called the *cost function* of x. What happens if we change the number of products produced? This gives us a rate of change! We get the following:

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$$

which tells us how much everything is changing. The *marginal cost* is just the instantaneous rate of change of the cost. In other words

$$\text{marginal cost } = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

Wait!!! Hold up. You can't have half a product. Like if I'm selling a doll, I can't just sell half a doll! So what does $\Delta x \to 0$ even mean?! Basically what we are saying here is that if the number of products is big enough then we can appoximate the addition of one additional product, by taking the derivative. In other words we are saying:

$$C'(n) \approx C(n+1) - C(n)$$

This makes life easier since maybe calculating C twice is much more difficult for a computer (or a human) than calculating the derivative once and calculating it once.

Let's pretend we are working for Willy Wonka and he wants to figure out the marginal cost of the ever-lasting gobstopper. He's super secretive so he only wants to produce 10 of them for now. What is Wonka's marginal cost if $C(x) = \text{arcsec}(x) + x^2$.

First we must find the derivative!

$$C'(x) = \underline{\hspace{6cm}}$$

Then calculating at $x = 10$ we get:

$$C'(10) = \underline{\hspace{8cm}}$$

8.2 Exponential Growh and Decay - §3.8

We've also done some exponential growth and decay problems already. We saw this in the population growth case.

A lot of times, things grow and decay proportional to themselves. So in essence, in a population, you grow based on how many people are present, and that growth in measured by some proportion of the current number. As an example: If we take a census in 2015 and then take a census in 2020 then we might find that the population grew by 20%. What this means is that if P is the population in 2015, then $1.2P$ is the population in 2020. So the population is measured in proportions.

This rate of change can be made instantaneous by taking the derivative, giving us:

$$\frac{dy}{dt} = ky$$

where k is the proportion.

We've only seen one function where y stays itself after taking the derivative! The exponential!!! That means, usually with growth and decay, we are dealing with the exponential function.

In fact, it is known that the only solutions to the equation $\frac{dy}{dt}ky$ are the exponential functions:

$$y(t) = y(0)e^{kt}$$

Let's look at some examples where we use these exponential functions.

Example 8.3 Let's first look at the nucleur armageddon. In other words: radioactive decay. This example is different from the population growth one because we are actually decreasing the volume of a radioactive substance rather than increasing. In essence, if we have some particle then it's mass decays exponentially by the following formula:

$$m(t) = m_0 e^{kt}$$

where m_0 is the initial value, k is some constant for the particle, and t is the variable. In order to calculate this, physicists use something called the half-life of a particle. The *half-life* of a particle is the time needed for half of the particle to decay. We use all these facts in the following example.

So let's suppose we have $1kg$ of uranium-235. It takes 704 million years for 1 gram of uranium-235 to decay to 1/2 a gram. In other words, the half-life of uranium-235 is 704 million years = 704000000. We already know that we are starting off with 1000 grams of uranium-235 so that means $m_0 = 1000$. But how do we find k?

Using the half-life! We already know that $m(704000000)$ should be half the initial mass since 704000000 is the half-life. So So

$m(t) =$ ______________________________

How much of our uranium will be left over in 5 billion years when the sun dies out?

Example 8.4 Our next example uses Newton's law of cooling. We're going to be looking at the temperature of an object over some time, so we denote this function by $T(t)$. Newton's law of cooling says that

$$\frac{dT}{dt} = k(T - T_s)$$

where T_s is the temperature of the surroundings. To calculate things, we need to fix things up. So we let $y(t) = T(t) - T_s$. Then $y'(t) = T'(t)$ and so our equation above becomes:

$$y'(t) = ky$$

which matches our exponential formula from above.

So suppose that we just made some hot boiling tea (or coffee) and it's just chilaxing at room temperature (20°). We know that in 30 minutes, the tea will be 55°. Considering that the perfect drinking temperature for tea is 65°, when should you drink your tea?

This is a ton of information! So let's break it down slow. First, let's figure out our formula:

$$\frac{dT}{dt} = k(T - T_s) = \underline{\hspace{6cm}}$$

We also know that $T(0) = \underline{\hspace{2cm}}$ and so $y(0) = T(0) - T_s = \underline{\hspace{8cm}}$.
Also, we know that $T(30) = \underline{\hspace{1.5cm}}$ and so $y(30) = T(30) - T_s = \underline{\hspace{7.5cm}}$.

Then $y(t) = y(0)e^{kt}$ (from our equation earlier!). We can now calculate k:

So now we want to figure out when $T(t) =$ ____ in other words when $y(t) =$ ______________________. Therefore: In other words, we must wait about 20

minutes after pouring tea before it is at perfect temperature.

8.3 Related Rates - §3.9

Related rates are notoriously complicated. They are word problems that are difficult to understand and don't relate to anything in the real world. Although they are not fun, they teach us how to problem solve in the real world. At a job, no one is ever going to give you a function and say "find its derivative". Instead they are going to explain a problem to you, give you a ton of data, and expect you to solve it. This is what this section aims to do:

(1) ____

(2) ______________________________________

(3) ____

These are hard because they require a lot more thinking than just memorization. Even though they are difficult, there are some steps you can take to help you:

(1) __________________________________

(2) __________________________________

(3) ________________________

(4) ____

(5) ____

(6) ____

(7) ______

Example 8.5 You're walking along the street one day when and you remember it's your friend's birthday tomorrow! You totally forgot to by them a birthday present!!! But then you come up with a brilliant idea, you're going to buy them a ton of balloons with their face on each one. You decide to go to Sky's: your local non-binary balloon seller! After a 30 minute walk, you get to Sky's and ask them about making you balloons with your friend's picture on it. Sky turns to you and says they would love to help, but their boss came up with a new rule. All employees need to be able to calculate the rate at which the radius of a balloon increases or else they can't sell any balloons. They turn to you and ask if you would help them figure it out. Armed with calculus, you tell Sky yes, and buckle down to help them.

Sky tells you that when the put the balloon on the helium pump, the spherical balloon's volume increases at 100 cm^3/s. Can you help them figure out the rate at which the radius of the baloon is increasing? What's the rate when the diameter is equal to 20 cm?

Example 8.6 After grabbing your balloons you continue on your way home. As you're heading home you decide to walk by a construction site to see your favorite construction worker: Sandra. You walk over to the construction site and see Sandra standing next to a ladder on the ground getting ready to do something. Wanting to know what's on her mind, you grab a hard hat from the table and walk over to her to see what's up. Not looking away from the ladder, she tells you she was just about to move the ladder to start painting the wall. She then turns to you and notices all your balloons and a grin appears on her face! "We should tie the balloons to the ladder and let the balloons lift the ladder up", she says. Although you don't want to lose the balloons, this seems like fun so you go with it. She takes the balloons and puts them on the 10 metre long ladder. You both start noticing that the ladder starts moving away from you both at a rate of 1 metres per second. How fast will the top of the ladder be moving away from you when the ladder is 6 metres up?

Exercise 8.7 After helping Sandra with her ladder, you realize you're gonna be late to your dinner date tonight! You run home, drop off the balloons and quickly freshen up. Looking at your watch you realize you're defs going to be late if you don't leave stat. You call an uber and decide to click on the "sports car driver" option. Within 30 seconds a sports car driver meets you in front of your house, introduces himself as Vito. Remembering that Vito was your city's four time reigning champion in sports car driving, you know you're gonna make it to your date on time. You hop in his car and you're on your way.

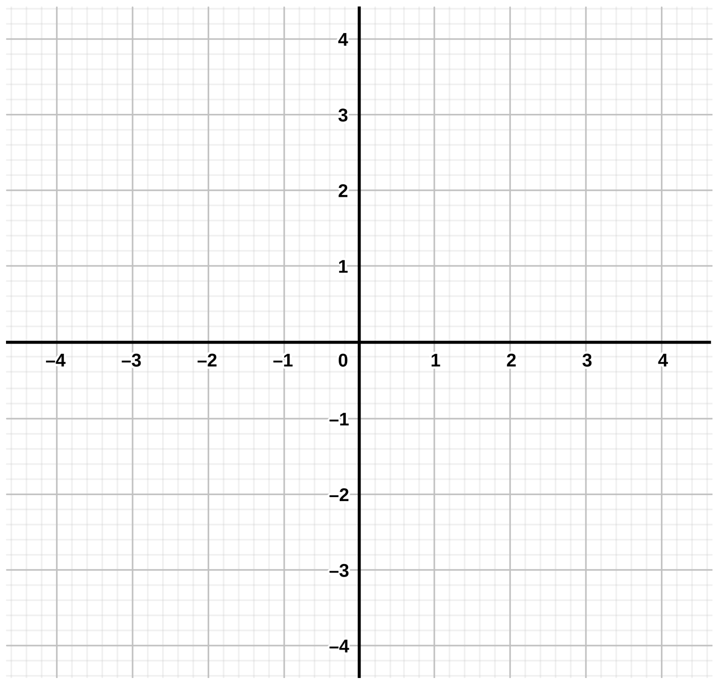
As you're heading to the restaurant, you look at Vito's speedometer and notice he's going 100 km/h! And in a school area! You're confident you'll make it in time, only to realize you're coming up to the "intersection of doom". You look around and notice another sports car uber driver on the other street approaching the intersection as well. The other driver is Morty! The infamous sports car driver from the city over who won nationals last year! Realizing that if your car doesn't hit the intersection first, you're going to be late to your date, you decide to quickly calculate who's going to arrive at the intersection first. Although you are 4 km from the intersection, you notice that the other car is going arond 80 km/h and that the distance between your cars is decreasing by 80 km/h. Will you make it to your date in time?

Example 8.8 Although you were late for your date, your date doesn't mind and you end up having an amazing time at the restaurant. So good in fact, that you invite them to your friend's birthday party tomorrow! When tomorrow comes, you decide you need to make a cake for your friend, but you want it to be a special cake, so you're going to make it a cone shaped cake! You look for your cone shaped pan and find the one with a 10 cm radius and is 20 cm tall. You make the cake mix and start pouring the batter into the pan (pointy side down) at around 2 cm^3/min. While your pouring, your date from last night calls and asks what you're doing before the party. You tell them and they ask you how fast the batter is rising in the pan? You notice that your batter just hit the 5 cm mark, how fast is it rising?

8.4 Linear approximations and differentials - §3.10

Recall that the derivative gives us the slope of the tangent line to a function. What if we want to get an approximation of the line itself and not just the slope?

Let a be some point on a graph.



The approximation $f(x) \approx f(a) + f'(a)(x-a)$ is called the *linear approximation* or *tangent line approximation* of f. The linear function whose graph is the tangent line is given by

$$L(x,a) = f(a) + f'(a)(x-a)$$

is called the *linearization of f at a.*

The book uses $L(x)$ instead of $L(x,a)$. This is confusing because the linearization depends on our choice of a! We'll see this in the next example.

Example 8.9 Let $f(x) =$ ____________________. Find the linearization of f at 1 and at 3.

First, let's find $f'(x)$.

(1) Let $a = 1$.

(2) Let $a = 3$.

Notice that these two linearizations are different! It's important to remember that a linearization depends on your choice of a.

How close is this approximation though?

Example 8.10 Suppose we want to calculate $f(3.112347)$ from the function above. In a calculator, we find:

$$f(3.112347) = 1.34966285$$

but sometimes, life happens and we don't have access to a calculator and/or the calculator is wrong. So let's use our linear approximation to see what the approximation would give. We said that $f(x) \approx L(x, a)$ for some a close to x. Let $a = 3$ since 3 and 3.112347 are fairly "close". Then

$$L(3.112347, 3) = \underline{\hspace{6cm}}$$

This is not to bad of an approximation!

The whole point of approximation is to make the calculation significantly easier by using a linear function to approximate, rather than trying to solve some big nasty complicated formula.

Example 8.11 Find a good approximation for $f(x) = x^x$ for $x = \frac{8}{9}$.

Exercise 8.12 With a partner, find the linearization of $f(x) = \sin(x)$ for x really close to 0.