Week 5

3–7 Feb 2020

5.1 Derivatives of polynomials - §3.1

All of the above is fun, but this can get really complicated, really quickly. So instead, let's try and generalize our results so that we don't have to work as hard.

From our previous examples we see that:

$$\frac{d}{dx}(x^2) = \underline{\hspace{2cm}} \qquad \frac{d}{dx}(x) = \underline{\hspace{2cm}}$$

We're tempted to state the following (which turns out to be true!)

Theorem 5.1 *If n is a real number, then*

Proof. We're actually gonna prove this one for positive integers, because it's nice and (relatively) easy. Two parts:

(1) ____
.

(2)

□

Example 5.2

- What is the derivative of $f(x) = x^{93}$? __________ .
- What is the derivative of $f(x) = \sqrt[5]{x^3}$? __________ .
- What is the derivative of $f(x) = x^{\pi}$? __________ .

Using our limit laws we have the following theorem:

Theorem 5.3 *Let f and g be differentiable functions and c a constant.*

- ______________________________
- ____
- ____

This means we can differentiate more complicated polynomials!

Example 5.4 What is the derivative of $f(x) = x^9 + 3x^4 - x^2 + 1$?

Example 5.5 Let's apply this to something real world. Suppose we have a rod (or a piece of wire) of metal. The rod is "homogeneous" if the density of the rod is linear throughout. In other words it is defined to be the mass (kg) per unit length (m). We can write this as ρ (density) is equal to the mass m divided by length ℓ:

$$\rho = \frac{m}{\ell}$$

But, now suppose our rod is not homogeneous. Instead, suppose that the mass is given in some function $m = f(x)$. Then, the average density is:

$$\rho = \frac{\Delta m}{\Delta \ell} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This looks exactly like the derivative! So then the density at any given point is given by:

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta \ell} = \frac{dm}{dx}$$

For example, let $f(x) = x^{3/2}$. Using the limit rules we just saw, we know $f'(x) =$ __________. In other words at any point x on our rod, $\rho =$ __________.

So if we look 1 metre up our rod, then we know the density is:

5.2 Exponential functions - §3.1

Let's talk about exponential functions and what their limits look like. Recall that an exponential function is a function of the form $f(x) = b^x$ for b a real number. Let's calculate its derivative.

$$\begin{aligned} f'(x) &= \underline{\hspace{5cm}} \\ &= \underline{\hspace{3.5cm}} \\ &= \underline{\hspace{3.5cm}} \\ &= \underline{\hspace{3.2cm}} \\ &= \underline{\hspace{4cm}} \\ &= b^x f'(0) \end{aligned}$$

But now we need to figure out what $f'(0)$ is! There is no easy way to do this, so we need to go back to our old methods of looking at multiple values of h.

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$
0.1	0.7177346...	1.1612317...
0.01	0.695555...	1.104669...
0.001	0.693387...	1.099216...
0.0001	0.693171...	1.0098673...

It turns out that

$$b = 2 \Rightarrow f'(0) = \lim_{x\to 0} \frac{2^x - 1}{x} = \underline{\hspace{2cm}}$$
$$b = 3 \Rightarrow f'(0) = \lim_{x\to 0} \frac{3^x - 1}{x} = \underline{\hspace{2cm}}$$

So, we have

$$\frac{d}{dx}2^x = \underline{\hspace{2.5cm}} \qquad \frac{d}{dx}3^x = \underline{\hspace{2.5cm}}$$

and in general:

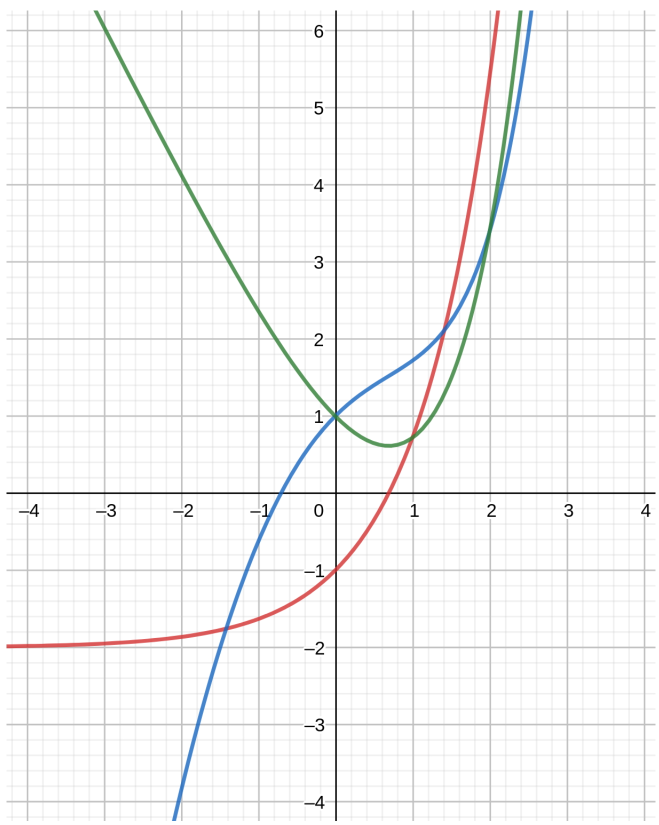
$$\frac{d}{dx}b^x = \underline{\hspace{2.5cm}}$$

What would be really cool is if we could find a number where $f'(0) = 1$. This would imply ____________________________________!! This is where the number __ comes from. If _________ then ____________________ and we are happy.

Example 5.6 Find the second derivative of $f(x) = e^x - x^2$.

$$f''(x) = e^x - 2$$

The graph of these functions is the following:



Example 5.7 Let's look at an example from biology. Suppose we're looking at the population growth of some animal over time. If we say that the number of animals at any given time is given by the function $n = f(t)$ then we can calculate the average rate of growth by the change in the number of animals over some given time:

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Notice how (again) this looks like the definition of the derivative!

If we want to then know what the rate of growth at any given time (aka the instantaneous rate of growth), then we just need to take the limit as the change in time goes to 0:

$$\text{instantaneous rate of growth} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Suppose our growth is given by $f(t) = 2e^t + 3t$. What is the instantaneous rate of growth?

We know from this section that $f'(t) =$ ____________. Therefore, if want to know how much the population is growing in exactly one year then:

5.3 Product and quotient rules - §3.2

Recall that when we did addition and subtraction of derivatives, we saw that a very similar thing happened as with the limit laws; we could just separate them. For the limit laws we had:

$$\lim_{x \to a} (f(x) + g(x)) = \left(\lim_{x \to a} f(x)\right) + \left(\lim_{x \to a} g(x)\right)$$

And for derivatives we had:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x).$$

We now ask the same question for multiplication since we have the same limit law:

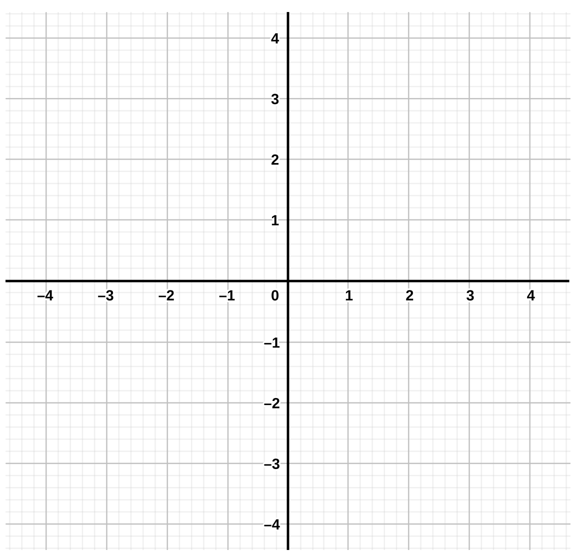
$$\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$

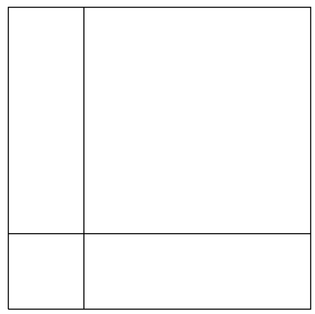
It turns out the same property does *not* hold for multiplication (and division)! In particular we have $(fg)' \neq f'g'$ like we would want.

Example 5.8 Let's look at a super simple example for proof. Let $f(x) = x$ and $g(x) = x$. Then,

So we need to figure something else out to calculate $(fg)'$. This is a bit complicated, but it turns out the best way is to use geometry.

First, let's look at one of our functions.





This is called the *product rule*: If f and g are both differentiable, then

Example 5.9 Let's do an example. Let $f(x) = x^2 e^x$. What is $f'(x)$?

We can also do quotients based off a similar technique as the product.

This is called the *quotient rule*: If f and g are both differentiable and $g(x) \neq 0$, then

Note that in the quotient rule, the *order* matters since we are subtracting. Make sure to keep this in mind.

Example 5.10 Let $f(x) = \frac{x^3+3x}{x^2+1}$.

Therefore,

$$f'(x) = \frac{x^4+3}{x^4+2x^2+1}$$

Exercise 5.11 With a partner find $f'(x)$ where $f(x) =$ ____________________.

Example 5.12 Our next real world example comes from electrical engineering. There's a law in EE called "Ohm's Law" which states that the voltage is equal to the current times the resistance at any given point in a circuit,*i.e.*, $V = IR$. We're going to try and calculate the instantaneous voltage at a given time if both the current and the resistance is changing over time.

If $I = i(t)$ is the function for the amount of current at any given time, then the average current over time is given by:

$$\text{average current } = \frac{\Delta I}{\Delta t}$$

and the instantaneous current at any given time is given by:

$$\text{current } = \lim_{\Delta t \to 0} \frac{\Delta I}{\Delta t} = \frac{dI}{dt} = i'(t)$$

Similarly, if $R = r(t)$ is the function for the resistance at any given time, then the average resistance over time is given by:

$$\text{average resistance } = \frac{\Delta R}{\Delta t}$$

and the instantaneous resistance at any given time is given by:

$$\text{resistance } = \lim_{\Delta t \to 0} \frac{\Delta R}{\Delta t} = \frac{dR}{dt} = r'(t)$$

We know that the average rate of voltage over time is given by:

$$\text{average voltage } = \frac{\Delta V}{\Delta t}$$

and, by what we saw earlier, the instantaneous voltage is given by:

$$\text{voltage } = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt} = \underline{\hspace{6cm}}$$

Let's take as an example $I = i(t) = t^2$ and $R = r(t) = e^t$. Then what is the instantaneous voltage when $t = 2$?