Week 4

27–31 Jan 2020

Based off last week, what do you think the following theorem should say?

Theorem 4.1 *If $r > 0$ is a rational number such that x^r is defined for all x, then*

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = \_\_$$

This theorem is ridiculously strong! Let's look at an example.

Example 4.2 Let $f(x) =$ ______________________________. What is

$$\lim_{x\to\infty} f(x) = \underline{\quad}?$$

Example 4.3 Evaluate

$$\lim_{x \to \infty} 2x - \sqrt{4x^2 + 1}.$$

Example 4.4 Evaluate

$$\lim_{x \to \infty} \cos(x)$$

Example 4.5 Evaluate

$$\lim_{x \to \infty} x$$

Example 4.6 Evaluate

$$\lim_{x \to \infty} x^3 - x^2$$

Exercise 4.7 With a partner try and evaluate

$$\lim_{x \to \infty} \frac{x^2 - x}{2x - 5}$$

Definition 4.8 (Precise definition of limit at infinity) Let f be a function defined on the interval (a, ∞) for some number a. Then

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

Let f be a function defined on the interval $(-\infty, a)$ for some number a. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

Let f be a function defined on the interval (a, ∞) for some number a. Then

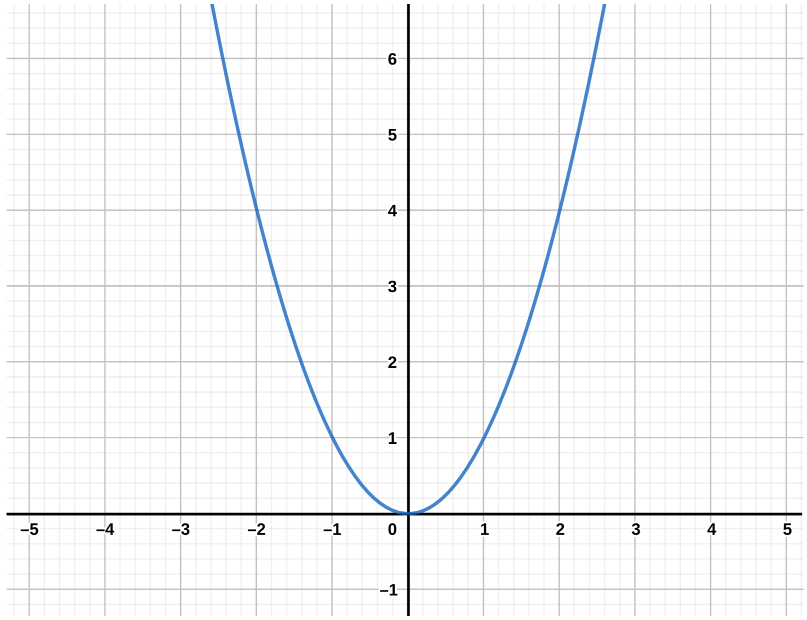
$$\lim_{x\to\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

4.1 Derivatives - §2.7

Remember how at the beginning of limits we looked at a tangent line to a curve. We can now use all the technology we've developed to answer this question for any arbitrary curve!

Let's look at that very first exercise. Let $f(x) = x^2$, what is the slope of the tangent line at $x = 1$?



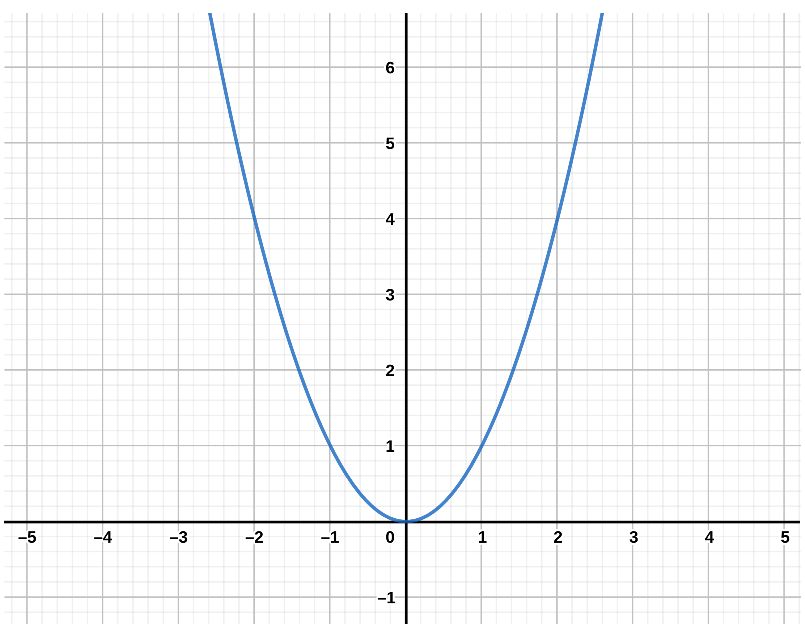
Recall that the method we used was to find secants to the curve. So we used equations like:

If we keep moving x closer and closer to a then we will eventually find the tangent line. In other words, the tangent line at point $(a, f(a))$ has as slope:

and from there you can figure out the tangent line for any given point!

Exercise 4.9 Use this method to find the tangent line $y = mx+b$ when $a =$ __ with a partner.

A lot of books (like ours) find the slope in another method. They fix a distance away from the point a and they try and decrease this distance. What does this mean?



Start with some point a where you want to find the tangent line. You let h be some distance away from the point a. Then you try and decrease the distance and solve the following limit:

Sometimes, this method is easier to solve than the previous version.

Example 4.10 Let $f(x) = 1/x^2$ and find the slope of the tangent line at $(2, \frac{1}{4})$.

This slope line has a special name and it is (kinda) the whole point of this course.

Definition 4.11 The *derivative of a function f at a number a* is

4.2 Rates of change - §2.7

We now are going to talk about some applications of the derivative. In particular, we're going to talk about rates of change.

We'll start off today on a rocket ship. You are tasked with helping figure out whether or not, at any point, the acceleration of the rocket is to much for the human body to handle (we don't want to kill off the astronauts!) A g-force is calculated at $9.8\frac{m}{s^2}$ and your scientists say that you don't want more than 3g for an astronaut. In other words, your acceleration can't go above $3 \cdot 9.8 = 29.4$.

You go and talk with the pilot and they say it will take roughly 120 minutes to reach escape velocity (the velocity needed for you to escape the earth's atmosphere) and that your velocity is given by the function $f(x) =$ _____.

Will the astronauts survive?

What we need to do is calculate the acceleration of the astronauts/space shuttle. Acceleration is just the rate at which something changes velocity. So what we need to do is calculate the rate of change!

Definition 4.12 Let $y = f(x)$ be a function. The change in x from x_1 to x_2 is given by

The change in y from x_1 to x_2 is given by

The *average rate of change* is the quotient of these two changes:

This probably looks familiar! But we need to know whether acceleration at any given moment is going to exceed 29.8, not just the average rate of change! The limit will give us the acceleration.

Definition 4.13 The *instantaneous rate of change* of a function $y = f(x)$ is given by

The instantaneous rate of change of velocity is acceleration.

So let's see if our astronauts will survive.

There are tons of examples of rate of change in real life. You can look at the book for more examples or you can do a google search. The instantaneous rate of change (aka the derivative) is the most crucial part of this and so we'll start exploring the derivative even further.

4.3 The derivative - §2.8

Recall that we've so far been looking at the derivative of a function f at a fixed number a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

What if we want to look at the derivative at an arbitrary point, not just at a? For this we replace the number a by a variable.

We already did this when we were looking at acceleration. Notice that this allows us to look at f' as a function! This function, f', is called the *derivative of f*.

Example 4.14 Let's do a test run with a super easy example. Find the derivative of $f(x) = x$.

Therefore $f'(x) = 1$.

Exercise 4.15 Try a more complicated example with a partner. Find the derivative of $f(x) = x^2 - x$.

There are a lot of different ways that people write the derivative $f'(x)$. This comes from the history of derivation!

$$f'(x) = \dot{y} = \frac{dy}{dx} = \frac{df}{dx} = Df$$

Definition 4.16 We say that a function f is *differentiable at a* if $f'(a)$ exists.

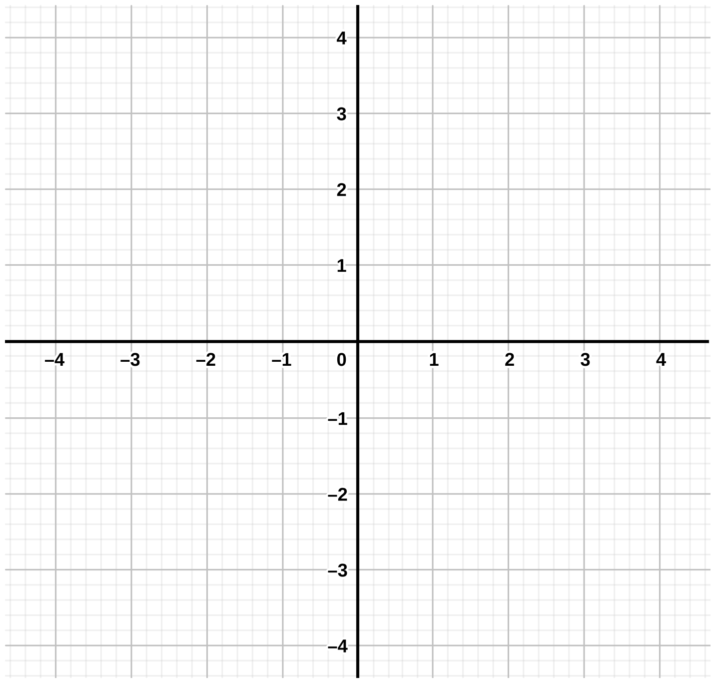
Example 4.17 Where is the following function differentiable?

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Being differntiable, actually tells us whether our function is continuous or not!

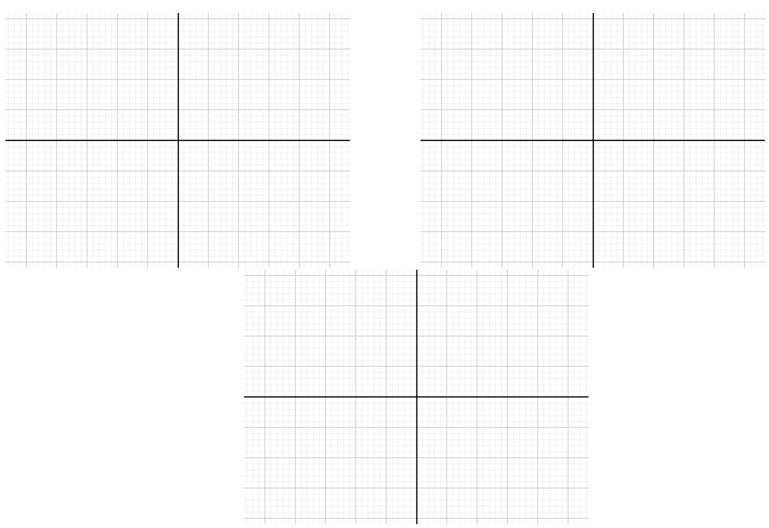
Theorem 4.18 *If f is differentiable at a, then f is continuous at a.*

We see this already in our example!



The opposite direction is *not* true! There are functions that are continuous at a point but not differentiable! There is an example of such a function in the book.

Intuitively, there are three main cases when a function is not differentiable:



4.4 Higher derivatives

Recall that when we were in outerspace that acceleration was calculated as the instantaneous rate of change (aka derivative) of the velocity. Velocity itself is also a rate of change! It is the instantaneous rate of change of position. So if velocity is the derivative of position, and acceleration is derivative of velocity then acceleration is the derivative of the derivative of position! There are two derivatives!

This happens because we're able to view the derivative as a function itself. And since it's a function, we can ask what is the derivative of the derivative. This is normally called the *second derivative*. In a similar fashion we can define the *third derivative*, *fourth derivative*, *nth derivative*, etc. Notation-wise, this is what they look like:

$$\begin{aligned}
2^{nd}: &\quad f''(x) = \ddot{y} = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = D^2f \\
3^{rd}: &\quad f'''(x) = \dddot{y} = \frac{d^3y}{dx^3} = \frac{d^3f}{dx^3} = D^3f \\
4^{th}: &\quad f''''(x) = f^{iv}(x) = f^{(4)}(x) = \ddddot{y} = \overset{4}{\dot{f}} = \frac{d^4y}{dx^4} = \frac{d^4f}{dx^4} = D^4f \\
n^{th}: &\quad f^{(n)}(x) = \overset{n}{\dot{f}} = \frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = D^nf
\end{aligned}$$

Exercise 4.19 With a partner, find the third derivative of $f(x) = x^2$.