Week 3

20–24 Jan 2020

Definition 3.1 (Left-hand and Right-hand limits) We say

We use the exact same format to prove things in this case.

Example 3.2 Prove that ____________________________ .

(1) **Analysis:**

(a)

(b)

(2) **Proof:**

(a) ____________________________

(b) ____

(c)

(3)

One last precise limit definition!

Definition 3.3 (Infinite limits) Let f be a function defined on some open interval that contains the number a, except potentially not defined at a itself.

Example 3.4 Prove __________________________________ .

(1) **Analysis:**

(a)

(b)

(2) **Proof:**

(a) __________________________ .

(b) ___________________________ .

(c)

(3) Therefore, by definition of the infinite limit:

3.1 Limit laws - §2.3

In a lot of cases, the best way to know the limit is to plug in the number directly.

Theorem 3.5 *If f is a polynomial or a rational function and a is in the domain of f, then*

$$\lim_{x \to a} f(x) = \underline{\hspace{2cm}}$$

If we know the limit to a few functions at a point, then we can use that information to figure out the limit of their combinations!

Theorem 3.6 *Suppose c is some constant and suppose the following limits exist:*

$$\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)$$

Then

(1)

$$\lim_{x \to a} (f(x) + g(x)) = \underline{\hspace{8cm}}$$

(2)

$$\lim_{x \to a} (f(x) - g(x)) = \underline{\hspace{8cm}}$$

(3)

$$\lim_{x \to a} (cf(x)) = \underline{\hspace{4cm}}$$

(4)

$$\lim_{x\to a} (f(x)g(x)) = \rule{8cm}{0.4pt}$$

(5)

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \rule{10cm}{0.4pt}$$

(6)

$$\lim_{x\to a} (f(x))^n = \rule{4.5cm}{0.4pt} \quad \textit{when } n \in \mathbb{Z}^+$$

(7)

$$\lim_{x\to a} \sqrt[n]{f(x)} = \rule{4cm}{0.4pt}$$

Another important rule helps us when one function is extremely similar to another function.

Theorem 3.7 *If $f(x) = g(x)$ when $x \neq a$ then, if $\lim_{x\to a} f(x)$ exists, then*

We saw an example of this in Example 2.13.

Two more useful properties of limits are given next.

Theorem 3.8 *If $f(x) \leq g(x)$ when x is near a (except possibly at a), and the limits of f and g both exist as x approaches a, then*

Theorem 3.9 (The squeeze theorem) *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{xtoa} g(x) = L$$

The squeeze theorem is a super difficult theorem to use, but it is sometimes the easiest method of finding a limit!

Example 3.10 Let

$$f(x) = \underline{\hspace{5cm}}.$$

Show that $\lim_{x\to 0} f(x) = 0$. It would be super nice if we could just use one of our limit rules, but we found out in Example 2.14 that $\lim_{x\to 0} \cos\left(\frac{\pi}{2x}\right)$ does not exist. So, instead we're going to have to use the squeeze theorem.

One thing to notice is that cos always ossilates between -1 and 1. In other words:

$$-1 \leq \cos\left(\frac{\pi}{2x}\right) \leq 1$$

Multiply all sides by x^3 and we have

Now we have something that looks like the squeeze theorem! Since both x^3 and $-x^3$ are polynomials and 0 is in their domains we have:

And, by the squeeze theorem (!) we know that

Exercise 3.11 With a partner, use the squeeze theorem to show that

3.2 Continuity - §2.5

Limits are super easy when we can just plug in a number to get the answer. When we can just plug in a number a into a function $f(x)$ in order to get the limit we say that the function is continuous at a.

Definition 3.12 A function f is *continuous at a* if

For this equation to hold there are three things that normally need to be proved:

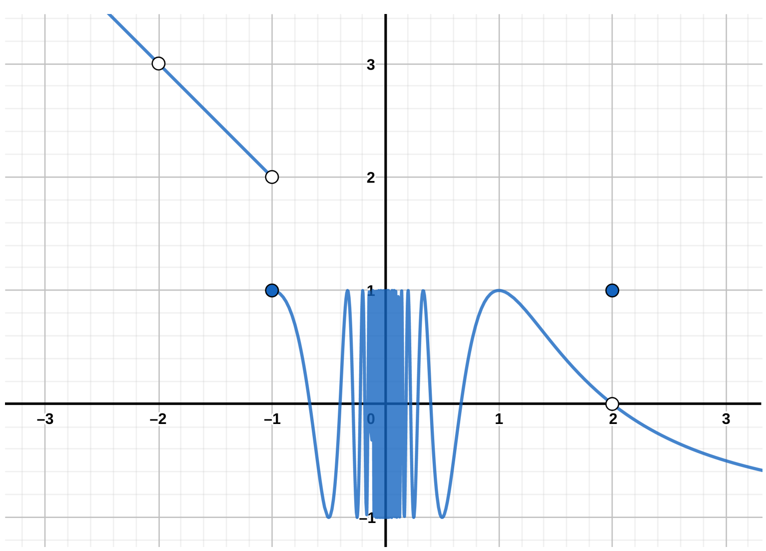
(1) ______________________________

(2) ______________________________________

(3) ____

If f is not continuous at a we say that it is *discontinuous at a*.

Example 3.13



We can similarly define left-hand and right-hand continuity.

Definition 3.14 A function f is *continuous from the left at a* if

A function f is *continuous from the right at a* if

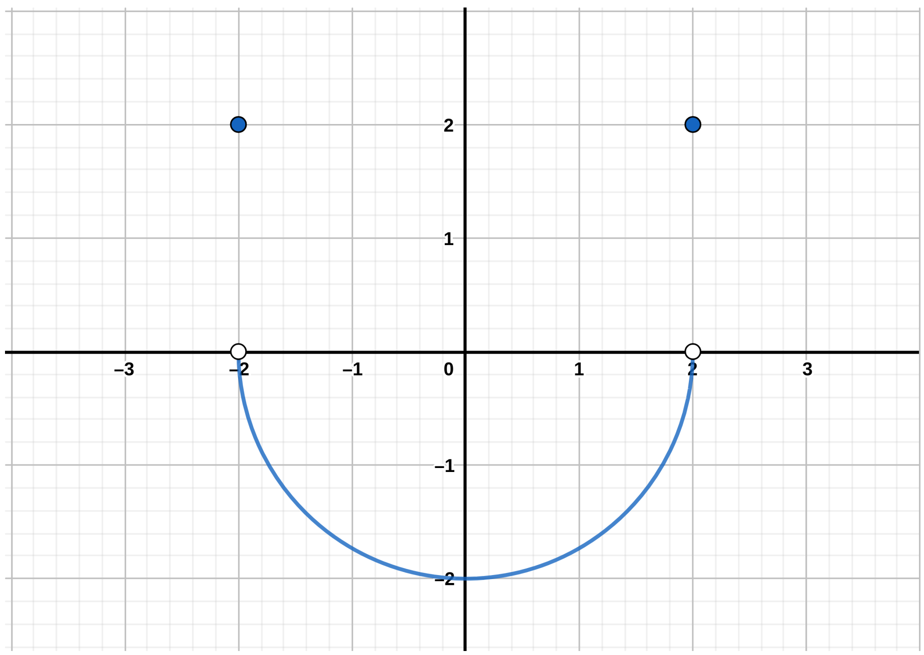
Similarly, we can do it on an interval!

Definition 3.15 A function f is *continuous on an interval I* if it is continuous at every number in the interval.

Example 3.16 Suppose

$f(x) =$

Show that f is continuous on the interval $(-2, 2)$, but discontinuous everywhere else. Here is a graph of the function:



Just like the limit laws, we have rules for continuity.

Theorem 3.17 *If f and g are continuous functions at a and c is some constant, then the following functions are also continuous at a:*

(1) __________

(2) __________

(3) ____

(4) _____

(5) ___

There are a lot of functions which are continuous!

(1) All ____________________ functions are continuous everywhere.

(2) _______________ functions are continuous on their domain.

(3) _______ functions are continuous on their domain.

(4) __________________ functions are continous on their domain.

(5) __________________ functions are continuous on their domain.

(6) __________________ functions are continuous on their domain.

Example 3.18 Where is the function $f(x) =$ ______________ continuous?

Exercise 3.19 With a partner, state where $f(x) =$ ________ is continuous?

What about composition $f \circ g$?

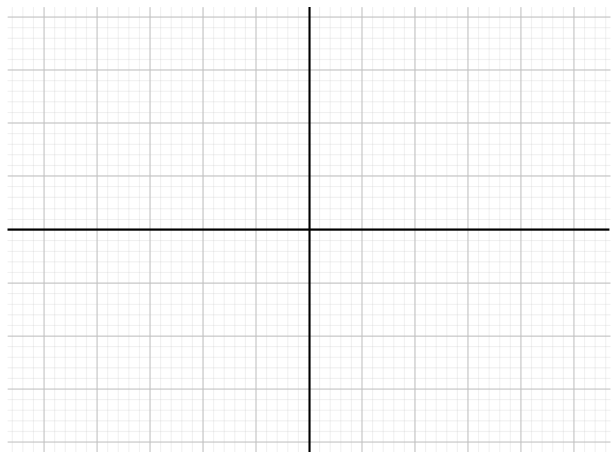
Theorem 3.20 *If f is continuous at b and $\lim_{x \to a} g(x) = b$, then*

If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

Continuous functions have a nice property in that they go through every number.

Theorem 3.21 (Intermediate value theorem) *If f is continuous on the interval $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = N$.*

This is best visualized through a picture.

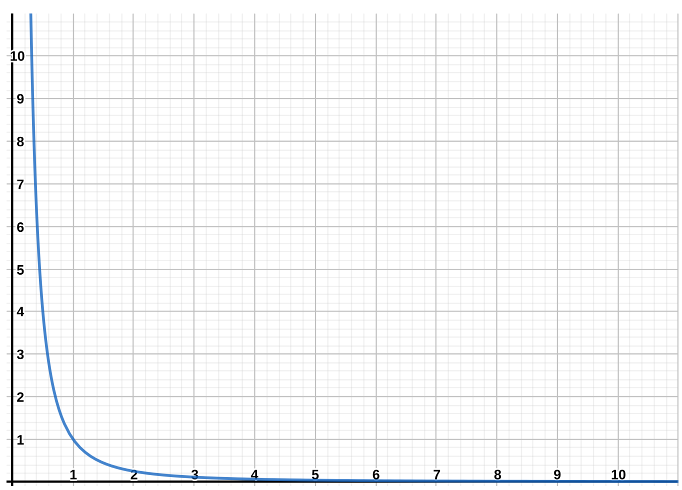


This theorem is super helpful for finding roots to polynomials!

3.3 To infinity and beyond - §2.6

Although infinity is not a number, we can look at limits as a function gets really really big, *i.e.*, when the function goes toward infinity.

Example 3.22 Let $f(x) = \_\_\_\_$ and notice that this function is always positive. Here is a graph of the function:



What this means is that the higher the value of x, the closer to __ we are going to get. Intuitively we say that the limit of $f(x)$ as x goes to infinity is 0.

We write this as:

We can play this game with any function. If f is a function defined on (a, ∞) then

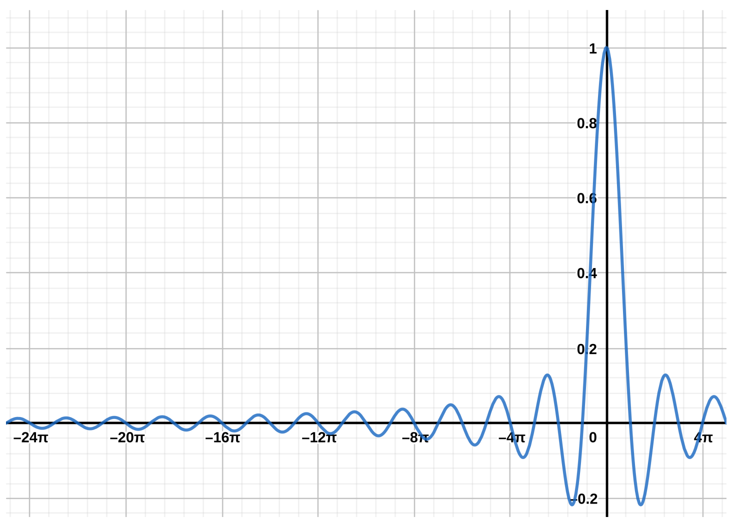
$$\lim_{x \to \infty} f(x) = L$$

means that the value of $f(x)$ approaches L as x gets larger and larger.

Similarly, you can do the same thing for negative infinity.

$$\lim_{x \to -\infty} f(x) = L$$

Example 3.23 Let $f(x) =$ _________ with the graph:



What is

$$\lim_{x \to -\infty} f(x) = \_\_$$

The line $y = L$ is called a *horizontal asymptote* of the curve $f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

There is a battle in mathematics on the definition of a horizontal asymptote! The majority give the definition above, but some authors additionally require that the line $f(x)$ doesn't cross the asymptote!

Exercise 3.24 Let $f(x) = \underline{\quad}$. With a partner, find

$$\lim_{x\to\infty} f(x) = \underline{\quad} \text{ and } \lim_{x\to-\infty} f(x) = \underline{\quad}$$