Week 2

13–17 Jan 2020

How do we find the inverse of a function?

(1) ________________________

(2) ____

(3) ____________________

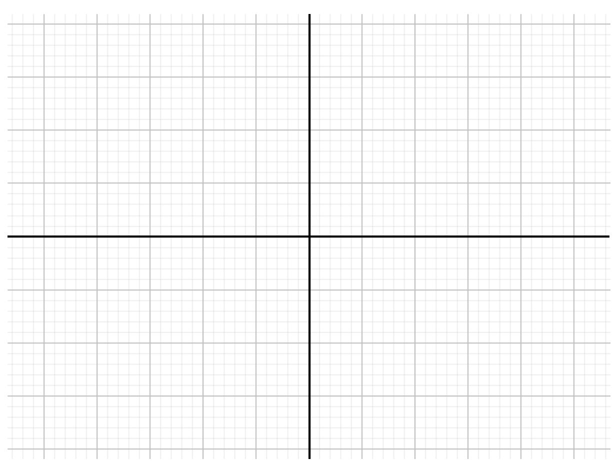
Example 2.1 If $f(x) =$ ____ then

(1) _________________________ .

(2) ___________ .

(3) _____________ .

Graphically, we can think of this as reflecting over the $y = x$ line.



Inverse and reciprocal are not always the same!

$$f^{-1}(x) \neq (f(x))^{-1}$$

Exercise 2.2 With the person next to you: Let $f(x) = \underline{\hspace{2.5cm}}$. What is its inverse function?

$$f^{-1}(x) = \underline{\hspace{3cm}}$$

What is its reciprocal function?

$$(f(x))^{-1} = \underline{\hspace{3cm}}$$

Another way to view this is by the *cancellation equations*:

There are two functions where finding the inverse becomes difficult: exponential and trigonometric functions. What are their inverses? For exponential, we let its inverse be called the *logarithm*. For $f(x) = b^x$ (where $b > 0$ and $b \neq 1$), the exponential passes the horizontal line test. The *logarithmic function with base* b, denoted $\underline{\hspace{1.5cm}}$, is the inverse of $f(x)$, *i.e.*, $f(x) = b^x$ implies $f^{-1}(x) = \underline{\hspace{3cm}}$.

Example 2.3 Let $f(x) = \underline{\hspace{1.5cm}}$.

Some nice properties of logarithms:

- $\log_b(xy) = \underline{\hspace{7cm}}$
- $\log_b(\frac{x}{y}) = \underline{\hspace{7cm}}$
- $\log_b(x^y) = \underline{\hspace{3.5cm}}$

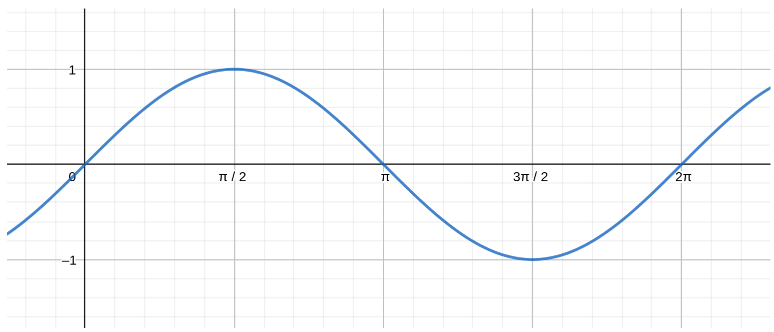
When $b =$ __, the logarithm is called the *natural logarithm* and is denoted __________ .

Some books will write $\log(x)$ (no subscript) to mean either $\log_{10}(x)$ or $\log_e(x) = \ln(x)$. Always double check with the book to ensure you know which version it is.

Trig functions are significantly more difficult. What is the inverse of the sine function?

Recall that you can just reflect over the __________ line to find the inverse. So let's do that.

Exercise 2.4 Reflect $f(x) = \sin(x)$ over the __________ line.



Turn to your neighbor and compare your drawing. Does this reflection give us an inverse function? _____ . How do we get an inverse function?

The inverse function for sine is known as the *arcsine* function and is the function:

Example 2.5 What is $\arcsin(\frac{\sqrt{3}}{2})$? ______

We can similarly do the same thing for cosine! The inverse function for cosine is known as the *arccosine* function.

Example 2.6 What is $\arccos(\frac{1}{2})$? ______

Exercise 2.7 In groups of 3–5, solve the following:

(1) $\arcsin(\frac{1}{2}) =$ ______

(2) $\arccos(\frac{\sqrt{3}}{2}) =$ ______

(3) $\arcsin(1) =$ ___

(4) $\arccos(1) =$ __

The inverse function for tangent is known as the *arctangent* function.

There are 3 other trigonometric functions:

$$\csc = \frac{1}{\sin} \qquad \sec = \frac{1}{\cos} \qquad \text{cotan} = \frac{1}{\tan}$$

Exercise 2.8 In the same group, solve the following:

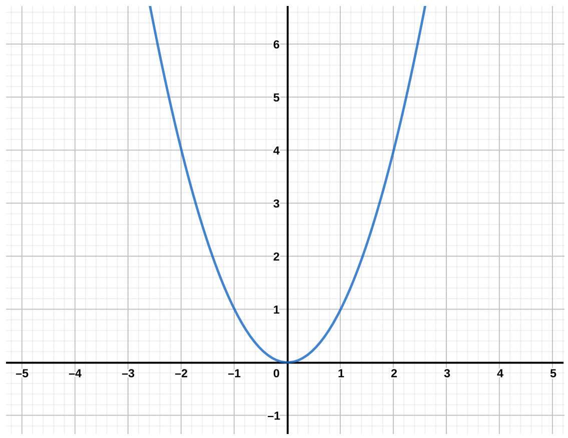
(1) $\arctan(1) =$ ______.

(2) $\arctan(0) =$ __.

(3) $\text{arccsc}(\sqrt{2}) =$ ______.

2.1 A tangential problem - §2.1

Exercise 2.9 Suppose we have the function $f(x) = x^2$. How do we find the slope of the tangent line at $x = 1$?



With a partner, find the slope of the tangent line at

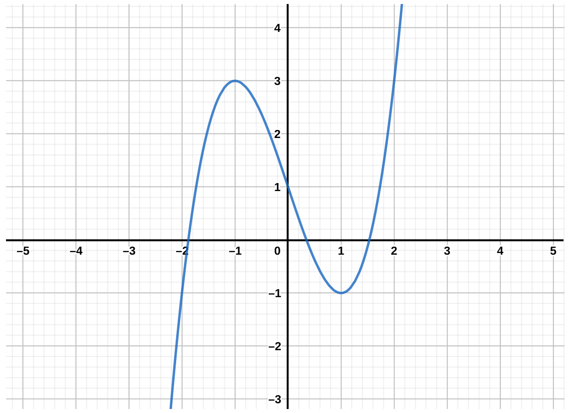
- $x = 0$
- $x = 2$
- $x = -1$

4hiddenSpace4cm Can you and your partner come up with a formula for the slope of the tangent line for arbitrary x?

2.2 Limits - §2.2

What we just did can be formalized into what's called limits. We want to find the value of a function at a certain point without knowing the value at that point exactly.

Example 2.10 Let's first start with an example to help clarify. Let $f(x) =$ ____________________.

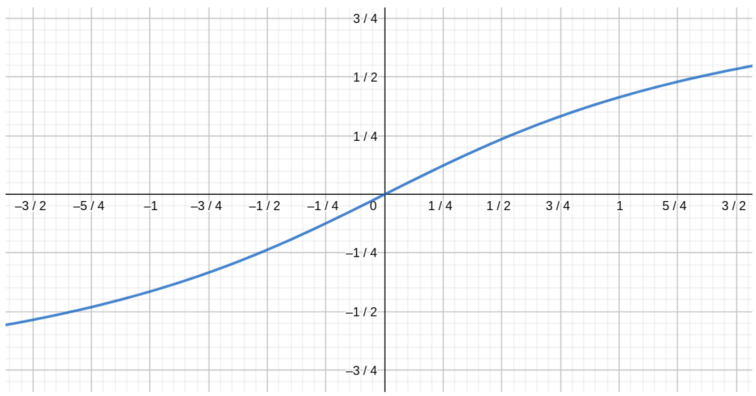


Try and calculate $f(\underline{\quad})$ without plugging in the number.

x	f(x)	x	f(x)
1	___	2	__
1.4	____________	1.6	__________
1.49	___________________	1.51	___________________
1.499	_____________________	1.501	__________________________
1.4999	__________________________	1.5001	_________________________________
1.49999	______________________________	1.50001	___________________________________

It looks like the value is approaching ________________________ . We write this as:

Exercise 2.11 With a partner find the limit of the function $f(x) =$ ____________ as x approaches ___ . (Use a calculator!) Here's the graph:



x	f(x)	x	f(x)
-1	________	1	________
-0.5	________	0.5	________
-0.1	________	0.1	________
-0.01	________	0.01	________
-0.001	________	0.001	________
-0.0001	________	0.0001	________

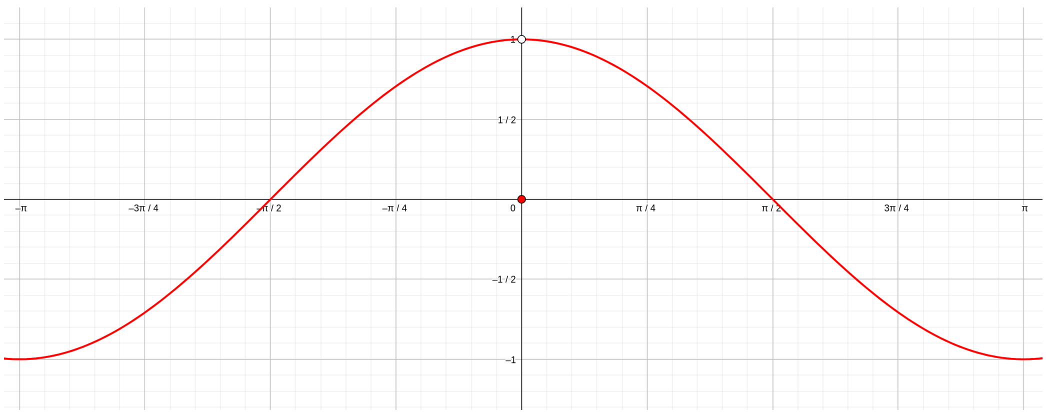
What do you both think is the limit?

Exercise 2.12 Again with your partner, try and find the limit of $f(x) =$ ________ at $x =$ ___.

Example 2.13 Suppose we have the following piecewise function

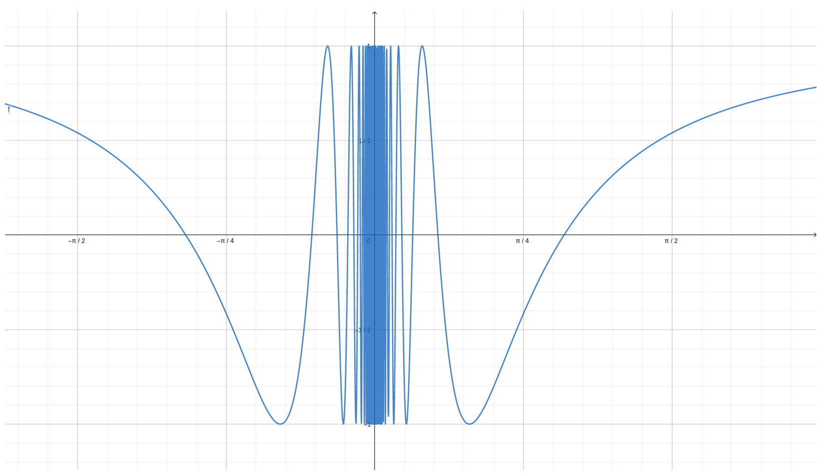
$$f(x) = \underline{\hspace{12em}}$$

and we want to know the limit at $x = 0$. Using the graph of the function, what do you think the limit is?



$$\lim_{x\to 0} f(x) = \underline{\quad}$$

Example 2.14 Let's look at something a little more complex. Say we have the function $f(x) = \underline{\hspace{8em}}$ whose graph is the following:



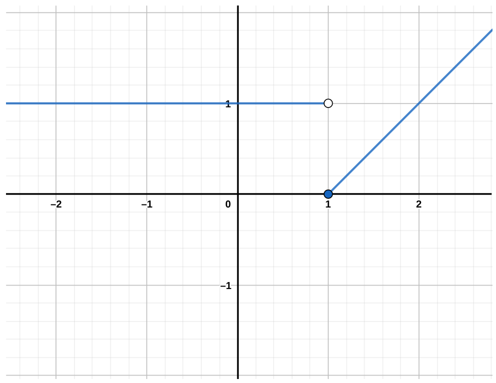
What is the limit at $x = 0$?

$$\lim_{x\to 0} f(x) = \underline{\hspace{12em}}$$

What happens if our function is split in two? Let

$$f(x) = \underline{\hspace{8cm}}$$

whose graph is given by:

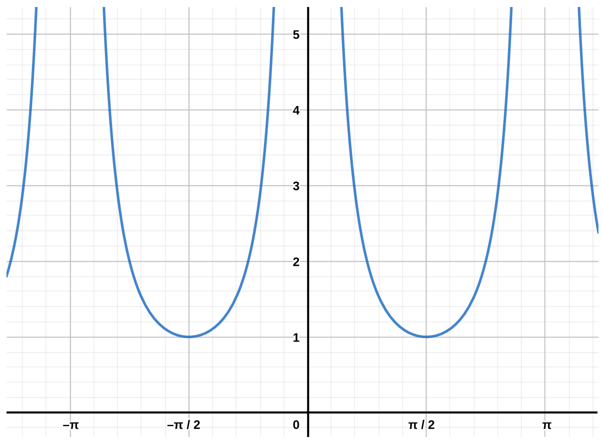


How can we calculate the limit at $x = 1$? What we can do is look at the limit from only one side instead of both sides like we had been doing. There are two sides of $x = 1$, a positive side and a negative side. Looking at the limit from the positive side, it looks like the limit is equal to __ . Looking at the limit from the negative side, it looks like the limit is equal to __ . We denote these by

Comparing these limits with our original definition of limit, we see

$\lim\limits_{x \to a} f(x) = L$ if and only if ____________________________________ .

Let's look at another example. Suppose $f(x) =$ __________ with the following graph:

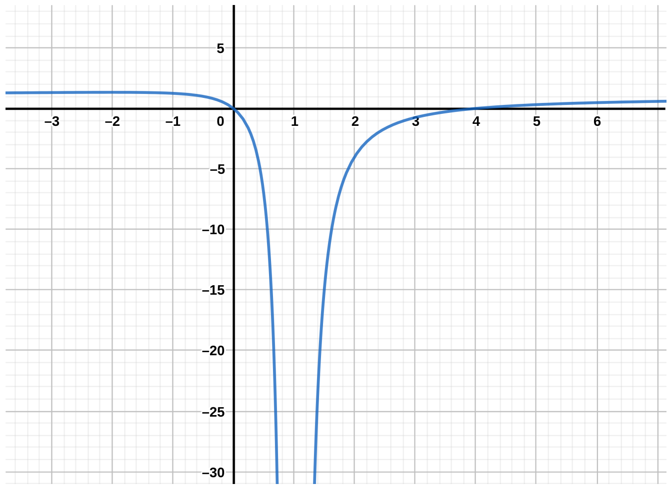


What is the limit as x approaches 0? The closer x gets to 0 (from either side) $\sin(x)$ gets closer to 0 and therefore $\frac{1}{\sin^2(x)}$ gets larger and larger. In fact, the limit does not approach any particular number, but keeps growing! To represent this, we say that the limit goes to infinity:

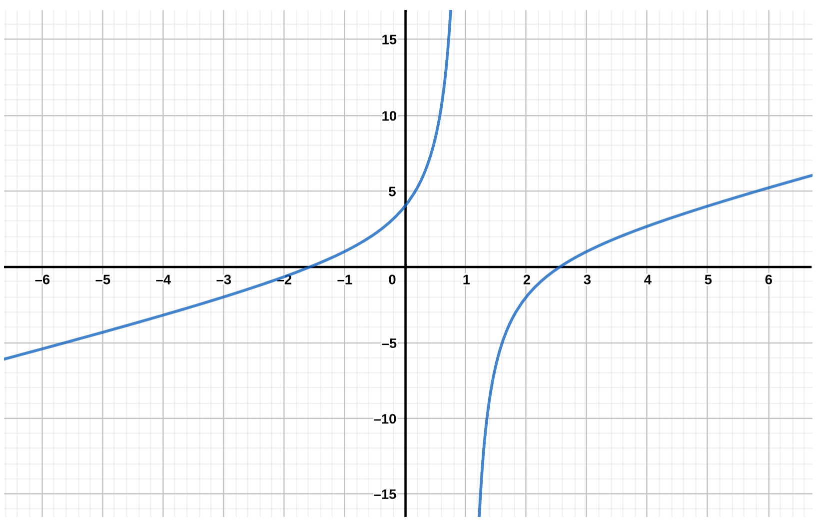
This does NOT mean that ∞ is a number! Nor are we saying that the limit exists. We are merely saying that as we approach $x = 0$, our function gets larger and larger and larger, without end.

Similarly, if the function $f(x)$ gets smaller and smaller as we approach a point a, then we can say that the limit is negative infinity:

Example 2.15 Let $f(x) =$ __________ then as $x \to 1$ the limit goes to negative infinity.



Exercise 2.16 With a partner, let $f(x) =$ ____________ whose graph is given by:



What are the following limits:

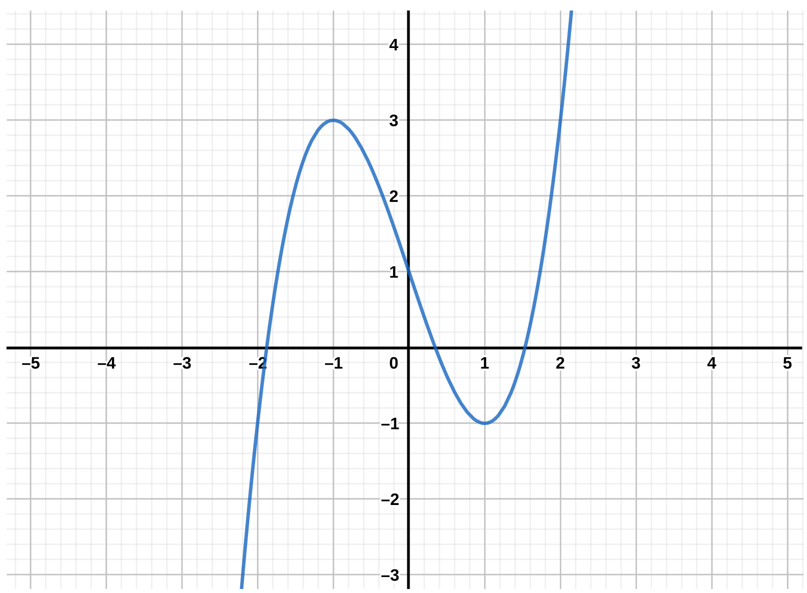
$$\lim_{x\to 1^-} f(x) = \underline{\hspace{2em}} \qquad \lim_{x\to 1^+} f(x) = \underline{\hspace{4em}} \qquad \lim_{x\to 1} = \underline{\hspace{5em}}$$

2.3 Precise definition of limit - §2.4

Let's be a little more precise by what we mean by a limit.

Definition 2.17 (Precise definition of a limit)

What does this mean?



We can use this definition as a way to show whether or not the intuitive solution to a limit problem is accurate or not. In order to prove that our intuition is correct, we use what's called an *epsilon-delta proof*. The following is a general outline of how an epsilon-delta proof works:

Epsilon-delta proof

(1) **Analysis:**

   (a) ____

   (b) ____

(2) **Proof:**

   (a) ____________________________

   (b) ____

   (c) __________________________________

(3) ____________________

Example 2.18 (Epsilon-delta proof) We want to prove that ______________________________ .

(1) **Analysis:**

(a) We start with $|f(x) - L| < \varepsilon$. In our case, $L =$ __ and $f(x) =$ __________. Therefore

(b) We want to end with something like ______________________ .

(2) **Proof:**

(a) Suppose ________ .

(b) Let _______________ .

(c) If _______________________ , then

(3) Therefore, by the definition of a limit

Exercise 2.19 With a partner, prove that ____________________________________
.

Example 2.20 Let's do a significantly harder example. Prove that __________________________________
.

(1) **Analysis:**

(a)

(b)

(2) **Proof:**

(a)

(b)

(c)

(3)