Week 13

30 Mar – 3 Apr 2020

13.1 The fundamental theorem of calculus - §5.3

Remember that in summations we had one summation which we called the telescoping sum:

$$\sum_{i=1}^{n} a_i - a_{i-1} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

In fact, our integral is written in almost the same way:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)(x_i - x_{i-1}).$$

So we now want to know if there exists an easier way to solve these integrals and we will see (very soon) that a way exists! In fact, this is done through antiderivatives!

Recall that an *antiderivative* of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Theorem 13.1 (The fundamental theorem of calculus) *If $f(x)$ is a continuous function on an interval $[a, b]$ and if $F(x)$ is an antiderivative of $f(x)$ then*

$$\int_a^b f(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

To make life easier, here are some functions with some antiderivatives

Function	**(an) antiderivative**
x^n where $n \neq -1$	$\frac{x^{n+1}}{n+1}$
x^{-1}	
$\sin(kx)$ where $k \neq 0$	$-\frac{\cos(kx)}{k}$
$\cos(kx)$ where $k \neq 0$	$\frac{\sin(kx)}{k}$
$\sec^2(kx)$ where $k \neq 0$	
$\operatorname{cosec}^2(kx)$ where $k \neq 0$	$-\frac{\operatorname{cotan}(kx)}{k}$
$\sec(kx)\tan(kx)$ where $k \neq 0$	$\frac{\sec(kx)}{k}$
$\operatorname{cosec}(kx)\operatorname{cotan}(kx)$ where $k \neq 0$	$-\frac{\operatorname{cosec}(kx)}{k}$
e^{kx} where $k \neq 0$	

Example 13.2 Let's look at some examples:

$$\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8-1}{3} = \frac{7}{3}$$

$$\int_2^4 2x\,dx = \underline{\hspace{10cm}}$$

$$\int_0^\pi \sin(x)\,dx = \underline{\hspace{3cm}}$$

$$\int_5^5 e^{x^2} \cdot \sin^2(x^2)\,dx = \underline{\hspace{0.5cm}}$$

Exercise 13.3 Try an example with a partner. Solve:

$$\int_1^3 4x^3 - e^x + \frac{1}{x}\,dx$$

Hint: Recall the properties of integrals from before.

13.2 Indefinite integrals - §5.4

Because of the fundamental theorem of calculus, we can find the integral $\int_a^b f(x)\,dx$. But in order to find this, we must first find an antiderivative of $f(x)$!

This is not always easy!

Example 13.4 Find a function $F(x)$ such that $F'(x) = f(x) = e^{x^2}$.

For this, we need to find a way to generate antiderivatives. This is normally done by using indefinite integrals.

A *indefinite integral* is the set of ____ antiderivatives of a given function. In other words, it's the general formula $F(x) + c$ that we saw a few days ago. We will normally denote this by:

All this is saying is that $F(x) + c$ is the general formula for an antiderivative of $f(x)$.

Note: Definite and indefinite integrals are different!!! Definite integrals have lower/upper limits and give you a value. Indefinite integrals have no lower/upper limits and give you a function.

We already have a ton of indefinite integrals that we can create using the derivative rules we saw earlier this semester:

(1) $\int k\,dx = kx + c$

(2) $\int [k_1 f(x) + k_2 g(x)]\,dx = k_1 \int f(x)\,dx + k_2 \int g(x)\,dx$

(3) $\int x^n\,dx =$ ______________

(4) $\int \frac{1}{x}\,dx = \ln(|x|) + c$

(5) $\int e^x\,dx =$ __________

(6) $\int a^x\,dx = \frac{a^x}{\ln a} + c$ (where $1 \neq a > 0$)

(7) $\int \sin(x)\,dx =$ ____________________

(8) $\int \cos(x)\,dx =$ ________________

(9) $\int \sec(x)\,dx = \ln(|\sec(x) + \tan(x)|) + c$

(10) $\int \operatorname{cosec}(x)\,dx = \ln(|\operatorname{cosec}(x) - \operatorname{cotan}(x)|) + c$

(11) $\int \tan(x)\,dx = \ln(|\sec(x)|) + c$

(12) $\int \operatorname{cotan}(x)\,dx = \ln(|\sin(x)|) + c$

(13) $\int \sec^2(x)\,dx = \tan(x) + c$

(14) $\int \operatorname{cosec}^2(x)\,dx = -\operatorname{cotan}(x) + c$

(15) $\int \sec(x)\tan(x)\,dx = \sec(x) + c$

(16) $\int \operatorname{cosec}(x)\operatorname{cotan}(x)\,dx = -\operatorname{cosec}(x) + c$

(17) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c$

(18) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(|x|) + c$

(19) $\int \frac{1}{1+x^2}\,dx = \arctan(x) + c$

But who cares? We care because it gives us an easier way to find the definite integral when we want to find it.

Example 13.5 Evaluate the following expression.

$$\int_2^4 \tfrac{12}{x^3}\,dx \quad =$$

$$= \tfrac{9}{8}.$$

Exercise 13.6 Try and find the following limit with a partner.

$$\int_1^4 \frac{1}{\sqrt{x^3}}\,dx$$

Example 13.7 Let's try a more complicated example. Let's find $\int_0^\pi 3x^2 - e^x + \cos(x)\,dx$.

$$\int 3x^2 - e^x + \cos(x)\,dx = \rule{8cm}{0.4pt}$$

Therefore

$$\int_0^\pi 3x^2 - e^x + \cos(x)\,dx =$$

$$= \pi^3 - e^\pi + 1$$

13.3 Substituion - §5.5

We're now going to look at the chain rule, but for integration. So suppose we have some (indefinite) integral like: $\int (3x+2)^{7/9}\,dx$? And we're trying to find the antiderivative of $(3x+2)^{7/9}$.

So far we haven't really learned how to tackle something like this, but we can use the ideas of the chain rule (and implicit differentiation) to solve something like this. What this method is called is "substitution".

What we want is an integral that is easier to handle. In our case, we want to change our integral to be something like $\int u^{7/9}\,du$. Why? Because we already know how to do this function!

So if we set $u = 3x+2$, then we have $\int u^{7/9}\,dx$. But now we have a different problem. What do you think the problem is?

Well, let's see if we can figure it out! We're gonna use implicit differentiation:

$$u = 3x + 2 \Rightarrow \rule{4cm}{0.4pt}$$

Therefore

$$\begin{aligned}\int (3x+2)^{7/9}\,dx &= \underline{\hspace{6cm}}\\ &= \frac{1}{3}\int u^{7/9}\,du\\ &= \underline{\hspace{6.5cm}}\\ &= \frac{9}{3\cdot 16}u^{16/9}+c\\ &= \frac{3}{16}(3x+2)^{16/9}+c\end{aligned}$$

And done!

So what did we do?

(1) ____________________

(2) ____

(3) _________________________________

(4) _________________________

(5) _____________________________________

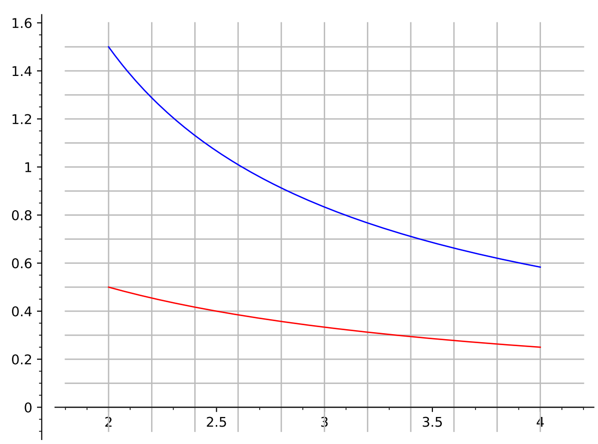
Exercise 13.8 With a partner, find the following integral: $\int x(7x^2-8)^{16/7}\,dx$.
Hint: Don't worry about the x by itself.

But recall that we're doing all of these things to find definite integrals! So what happens with that?

Example 13.9 Find the following definite integral: $\int_2^4 \frac{2x-1}{x^2-x}\,dx$ Therefore

But nowe we have a different problem!

Let's look at the two functions together. The one on top (blue) is the one with variable x and the one on the bottom (red) is the one with variable u.



So how do we solve this? We do exactly like we did with dx.

Here's the substitution rule in all its glory:

Theorem 13.10 (Substitution rule for a definite integral) *If $g'(x)$ is a continuous function on the interval $[a, b]$ and if $f(x)$ is a continuous function that has an antiderivative on the interval $[g(a), g(b)]$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \underline{\hspace{5cm}}$$

Where does the $g'(x)$ come from? From the chain rule!

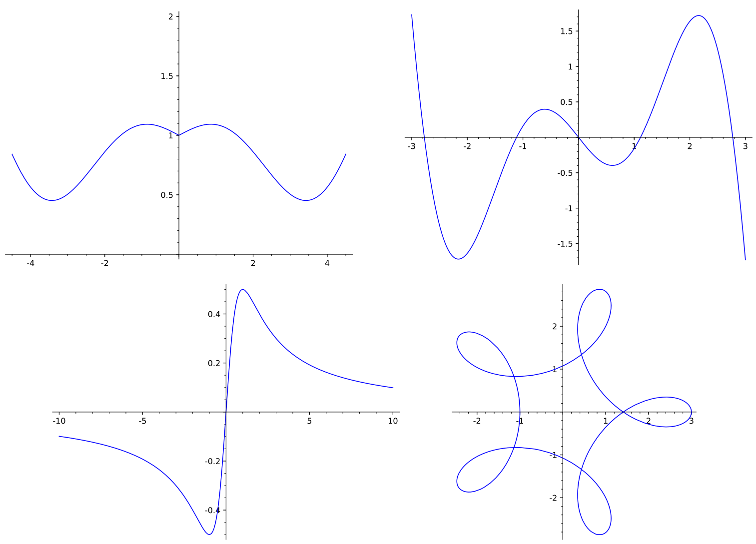
$$\begin{aligned} u = g(x) \Rightarrow\ & du = g'(x)\,dx \\ \Rightarrow\ & \frac{du}{g'(x)}\,dx \end{aligned}$$

and therefore to be able to change $f(g(x))$ to $f(u)$ we must always have $g'(x)$ there.

Example 13.11 Let's try this with a function that's a little bit more fun $\int \text{cotan}(ax)\, dx$ for $a \in \mathbb{R}$.

13.4 Even and odd functions - §5.5

Recall that a function is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. Some examples:



Theorem 13.12 (Properties of even and odd functions) *If $f_1(x)$ and $f_2(x)$ are even functions and if $g_1(x)$ and $g_2(x)$ are odd functinos, then, where the following operations are defined, we have:*

(1) $f_1(x)f_2(x)$ and $g_1(x)g_2(x)$ are ________ functions.

(2) $f_1(x)g_1(x)$ is an ______ function.

(3) $\int_{-a}^{a} f_1(x)\,dx =$ _______________________

(4) $\int_{-a}^{a} g_1(x)\,dx =$ __

Example 13.13 For example:

$$\int_{-a}^{a} x^2 \sin(x) - x^7 + x\cos(x)\,dx = \_\_$$