

Question 1 - Solution

$$\lim_{x \rightarrow 0} \sec(3x)^{\cot(4x)}$$

$$\sec(0)^{\cot(0)} = 1^\infty$$

$$\text{let } L = \lim_{x \rightarrow 0} \sec(3x)^{\cot(4x)}$$

$$\begin{aligned} \text{Then } \ln(L) &= \ln\left(\lim_{x \rightarrow 0} \sec(3x)^{\cot(4x)}\right) \\ &= \lim_{x \rightarrow 0} \left(\ln \sec(3x)^{\cot(4x)}\right) \end{aligned}$$

$$= \lim_{x \rightarrow 0} \cot(4x) \ln(\sec(3x))$$

$$= \lim_{x \rightarrow 0} \frac{\ln(\sec(3x))}{1/\cot(4x)}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(\sec(3x))}{\tan(4x)}$$

by L'Hôpital

$$= \lim_{x \rightarrow 0} \frac{3 \tan(3x)}{4 \sec^2(x)}$$

$$\frac{\ln(1)}{\tan(0)} = \frac{0}{0}$$

$\therefore$ we can use L'Hôpital's rule

$$\begin{aligned} \frac{d}{dx} \ln(\sec(3x)) &= \frac{3 \sec(3x) \tan(3x)}{\sec(3x)} \\ &= 3 \tan(3x) \\ \frac{d}{dx} (\tan(4x)) &= 4 \sec^2(x) \end{aligned}$$

$$= \frac{3}{4} \lim_{x \rightarrow 0} \frac{\tan(3x)}{\sec^2(x)}$$

$$= \frac{3}{4} \frac{\tan(3 \cdot 0)}{\sec^2(4 \cdot 0)}$$

$$= \frac{3}{4} \cdot \frac{0}{1} = 0$$

Since this fn is ok at $x=0$ we can plug in

$$\therefore \ln(L) = 0 \quad \& \quad e^{\ln(L)} = L = e^0 = 1$$

$$\therefore \boxed{\lim_{x \rightarrow 0} \sec(3x)^{\cot(4x)} = 1}$$

Question 2 - Solution

$$\lim_{x \rightarrow 0} \frac{(x+3)^4 x e^x}{e^{9x} - 1} \quad \frac{(0+3)^4 \cdot 0 \cdot e^0}{e^{9 \cdot 0} - 1} = \frac{0}{(-1)} = \frac{0}{0} \therefore \text{we can use L'Hôpital}$$

$$\text{by L'Hôpital} = \lim_{x \rightarrow 0} \frac{4(x+3)^3 x e^x + (x+3)^4 e^x + (x+3)^4 x e^x}{9e^{9x}}$$

$$= \frac{4(0+3)^3 \cdot 0 \cdot e^0 + (0+3)^4 e^0 + (0+3)^4 \cdot 0 \cdot e^0}{9e^{9 \cdot 0}} \quad \swarrow$$

$$= \frac{0 + 3^4 \cdot 1 + 0}{9e^0}$$

$$= \frac{3^4}{9}$$

$$= 3^2 = 3^2 = 9$$

$$= \boxed{9}$$

We can plug in since our f^n is continuous everywhere.

$$\frac{d}{dx}((x+3)^4 x e^x) = 4(x+3)^3 x e^x + (x+3)^4 e^x + (x+3)^4 x e^x$$

$$\frac{d}{dx}(e^{9x} - 1) = 9e^{9x}$$

Question 3 - Solution

$$\lim_{x \rightarrow 0} \frac{e^{x^4} - 1}{\cos(x^2) - 1} \quad \frac{e^0 - 1}{\cos(0) - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0} \therefore \text{we can use L'Hôpital's rule}$$

by L'Hôpital

$$= \lim_{x \rightarrow 0} \frac{4x^3 e^{x^4}}{-2x \sin(x^2)}$$

$$= \lim_{x \rightarrow 0} \frac{2x^2 e^{x^4}}{\sin(x^2)} \quad \frac{2 \cdot 0 \cdot e^0}{\sin(0)} = \frac{0}{0} \therefore \text{we can use L'Hôpital's rule.}$$

By L'Hôpital

$$= \lim_{x \rightarrow 0} \frac{4x e^{x^4} + 8x^5 e^{x^4}}{2x \cos(x^2)}$$

$$= \lim_{x \rightarrow 0} \frac{2e^{x^4} + 4x^4 e^{x^4}}{\cos(x^2)}$$

$$= \frac{2e^0 + 4 \cdot 0 \cdot e^0}{\cos(0^2)}$$

$$= \frac{2e^0 + 4 \cdot 0}{\cos(0)}$$

$$= \frac{2 \cdot 1 + 0}{1}$$

$$= \frac{2}{1} = \boxed{2}$$

Since $\cos$ everywhere is non-zero & $\cos(0) \neq 0$
 $\therefore$ we can plug-in

$$\frac{d}{dx} e^{x^4} - 1 = 4x^3 e^{x^4}$$

$$\frac{d}{dx} (\cos(x^2) - 1) = -2x \sin(x^2)$$

$$\frac{d}{dx} (2x^2 e^{x^4}) = 4x e^{x^4} + 8x^5 e^{x^4}$$

$$\frac{d}{dx} \sin(x^2) = 2x \cos(x^2)$$

Question 4 - Solution

$$y = \frac{(x^2 + 2)^{1/7} x^{1/5}}{(5x + 7)^5}$$

$$\ln(y) = \ln \left(\frac{(x^2 + 2)^{1/7} x^{1/5}}{(5x + 7)^5} \right)$$

$$= \frac{1}{7} \ln(x^2 + 2) + \frac{1}{5} \ln(x) - 5 \ln(5x + 7)$$

$$\frac{d}{dx} \ln(y) = \frac{d}{dx} \left(\frac{1}{7} \ln(x^2 + 2) + \frac{1}{5} \ln(x) - 5 \ln(5x + 7) \right)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{7} \frac{1}{x^2 + 2} \cdot 2x + \frac{1}{5x} - \frac{5}{5x + 7} \cdot 5$$

$$= \frac{2x}{7x^2 + 14} + \frac{1}{5x} - \frac{25}{5x + 7}$$

$$\frac{dy}{dx} = y \left(\frac{2x}{7x^2 + 14} + \frac{1}{5x} - \frac{25}{5x + 7} \right)$$

$$\boxed{\frac{dy}{dx} = \left(\frac{(x^2 + 2)^{1/7} x^{1/5}}{(5x + 7)^5} \right) \left(\frac{2x}{7x^2 + 14} + \frac{1}{5x} - \frac{25}{5x + 7} \right)}$$

Question 5: Solution

$$r^4 s^2 + \cos(rs)^2 + \sec(rs) = 3$$

$$\frac{d}{dr} (r^4 s^2 + \cos(rs)^2 + \sec(rs)) = \frac{d}{dr} 3 = 0$$

$$= 4r^3 s^2 + r^4 \frac{d}{dr} s^2 + 2 \cos(rs) (-\sin(rs)) \cdot \frac{d}{dr} (rs) + \sec(rs) \tan(rs) \frac{d}{dr} (rs)$$

$$0 = 4r^3 s^2 + r^4 \frac{d}{ds} s^2 \frac{ds}{dr} - 2 \cos(rs) \sin(rs) \left(s + r \frac{ds}{dr} \right) + \sec(rs) \tan(rs) \left(\frac{ds}{dr} + r \frac{ds}{dr} \right)$$

$$- 4r^3 s^2 = 2r^4 s \frac{ds}{dr} - 2 \cos(rs) \sin(rs) \cdot s - 2 \cos(rs) \sin(rs) \cdot r \frac{ds}{dr} + \sec(rs) \tan(rs) + \sec(rs) \tan(rs) \frac{ds}{dr}$$

$$2s \cos(rs) \sin(rs) - 4r^3 s^2 - \sec(rs) \tan(rs)$$

$$= \frac{ds}{dr} (2r^4 s - 2r \cos(rs) \sin(rs) + \sec(rs) \tan(rs))$$

$$\frac{ds}{dr} = \frac{2s \cos(rs) \sin(rs) - 4r^3 s^2 - \sec(rs) \tan(rs)}{2r^4 s - 2r \cos(rs) \sin(rs) + \sec(rs) \tan(rs)}$$

Bonus

$$\frac{d}{dq} (s^2) = \frac{d}{ds} s^2 \frac{ds}{dq}$$

$$= 2s \frac{ds}{dr} \frac{dr}{dq}$$

Since s is a f^n of r
& r is a f^n of q .

$$\frac{ds}{dq} = \frac{ds}{dr} \frac{dr}{dq}$$

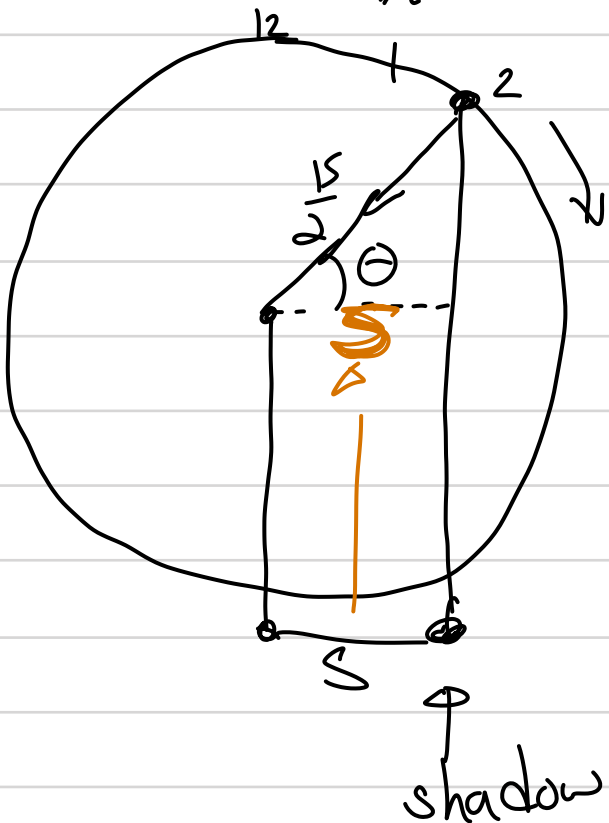
Question 6 - Solⁿ - Ferris Wheel

① Wheel

- Sun directly above
- 0.1 rev/min
- diameter is 15 meters
- 2'o-clock

What is speed of your shadow?

②



③

d = diameter

θ = angle

S = shadow position

r = radius

④

Solution

$$\cos(\theta) = \frac{S}{r}$$

$$d = 2r \Rightarrow r = \frac{d}{2}$$

⑤

$$\frac{d}{dt}(\cos(\theta)) = \frac{d}{dt} \frac{S}{r} \quad \leftarrow \text{constant}$$

$$-\sin(\theta) \frac{d\theta}{dt} = \frac{1}{r} \frac{dS}{dt}$$

⑥ $d = 15$

$$0.1 \frac{\text{rev}}{\text{min}} = \frac{2\pi}{10} \text{ m/min} = \frac{d\theta}{dt}$$

θ :

$\therefore \theta = \pi/6$

- as angle decreasing

$$\sin\left(\frac{\pi}{6}\right) \cdot \frac{2\pi}{10} = \frac{2}{15} \frac{dS}{dt}$$

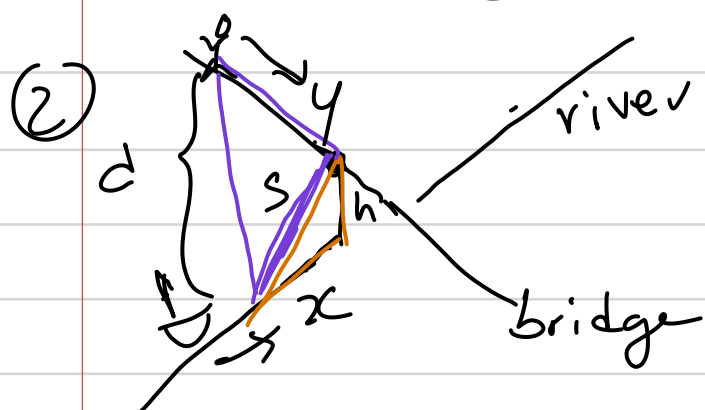
⑦

$$\frac{dS}{dt} = \frac{3}{2} \left(-\sin\left(\frac{\pi}{6}\right) \right) \frac{2\pi}{10} = 3\pi \left(\frac{1}{2} \right) = \boxed{\frac{3\pi}{2} \text{ m/min}}$$

Q 6 - Solⁿ Bridge

- 1. 2 meters above river
- right angle to bridge
- boat: 1 meter / second
- you are 7 meters from center when boat under bridge
- you: 1 meter / second

What rate is distance b/n you & boat changing?



- ③
- y: person's direction
 - x: boat's dir
 - h: height
 - d: distance b/n person & boat
 - s: distance b/n boat & bridge

④

$$s^2 = x^2 + h^2$$

$$d^2 = s^2 + y^2$$

$$d^2 = x^2 + h^2 + y^2$$

⑤ h is constant

$$\frac{d}{dt} d^2 = \frac{d}{dt} x^2 + \frac{d}{dt} h^2 + \frac{d}{dt} y^2$$

$$2d \frac{dd}{dt} = 2x \frac{dx}{dt} + 0 + 2y \frac{dy}{dt}$$

distance decreasing

⑥ $h=2, y=7, \frac{dx}{dt}=1, \frac{dy}{dt}=-1$, want to solve for $\frac{dd}{dt}$

$x=0$ (boat under bridge)

$$d^2 = x^2 + h^2 + y^2 = 0^2 + 2^2 + 7^2 = 4 + 49 = 53$$

$$d = \sqrt{53}$$

$$\therefore 2\sqrt{53} \frac{dd}{dt} = 2 \cdot 0 \cdot 1 + 0 + 2 \cdot 7 \cdot (-1)$$

⑦

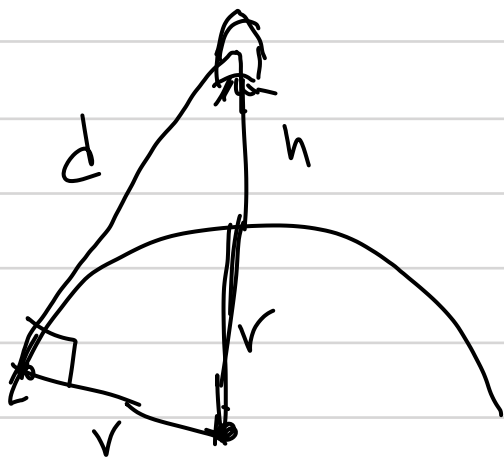
$$\frac{dd}{dt} = \frac{0 + 0 + (-14)}{2\sqrt{53}} = \boxed{-\frac{7}{\sqrt{53}} \text{ m/s}}$$

Q6 - Solⁿ - Space Shuttle

- (1) • going up
• 40 km/s
• radius of Earth: 4000 km
• Space shuttle 5000 km up

What rate is distance b/n Space shuttle & horizon changing?

(2)



- (3) r = radius of Earth
 h = height
 d = distance b/n horizon & space shuttle

(4) $(r+h)^2 = d^2 + r^2$
 $r^2 + 2rh + h^2 = d^2 + r^2$
 $2rh + h^2 = d^2$

(5) $\frac{d}{dt} d^2 = \frac{d}{dt} 2rh + h^2$

$2d \frac{dd}{dt} = 2r \frac{dh}{dt} + 2h \frac{dh}{dt}$ (r is constant)

(6) $h = 5000$
 $r = 4000$
 $\frac{dh}{dt} = 40$

$d^2 = 2rh + h^2 = 2(4000)(5000) + (5000)^2$
 $d = \sqrt{40\,000\,000 + 25\,000\,000}$
 $= \sqrt{65\,000\,000}$
 $= 1000 \sqrt{65}$

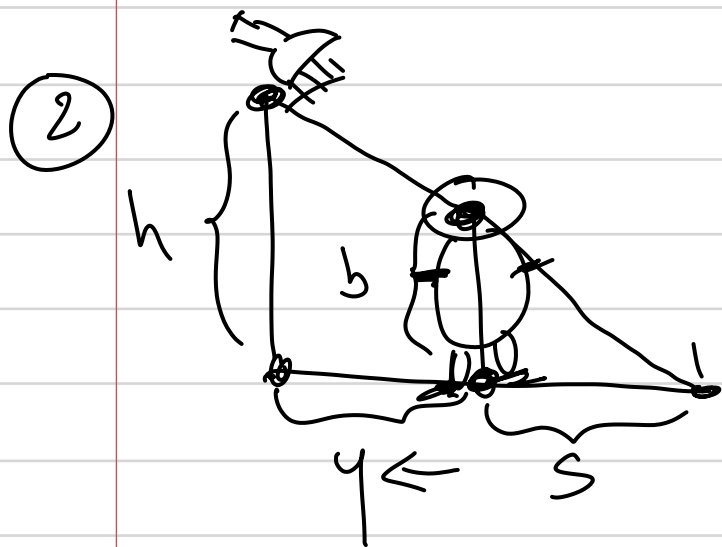
$2(1000 \sqrt{65}) \frac{dd}{dt} = 2(4000) \cdot 40 + 2(5000) \cdot 40$

(7) $\frac{dd}{dt} = \frac{2 \cdot 40 \cdot (4000 + 5000)}{2 \cdot 1000 \sqrt{65}} = \frac{9000}{25 \sqrt{65}} = \sqrt{\frac{360 \text{ km/s}}{\sqrt{65}}}$

Q6 - Solⁿ - Honey Bear

- ① Bear: 2 meter tall
 • flashlight: 4 meters up
 • Bear walking at 3 meters/second
 • Bear 8 meters away

How fast is length of shadow decreasing?



- ③ s = shadow length
 y = ground distance to light/hung
 h = height
 b = bear height

④ Similar triangles:

⑤ b, h are constant

$$\frac{b}{s} = \frac{h}{y+s} \Rightarrow by + bs = hs$$

$$\frac{d}{dt} by + \frac{d}{dt} bs = \frac{d}{dt} hs$$

$$b \frac{dy}{dt} + b \frac{ds}{dt} = h \frac{ds}{dt}$$

⑥ $h = 4, b = 2, \frac{dy}{dt} = -3$ (decreasing distance)
 $y = 8$

$$2 \cdot (-3) + 2 \cdot \frac{ds}{dt} = 4 \cdot \frac{ds}{dt}$$

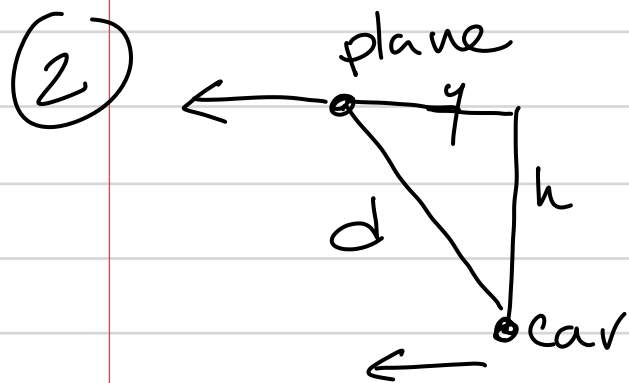
⑦

$$4 \frac{ds}{dt} - 2 \frac{ds}{dt} = -6 \Rightarrow 2 \frac{ds}{dt} = -6 \Rightarrow \frac{ds}{dt} = \frac{-6}{2} = \boxed{-3 \text{ m/s}}$$

Q6 - Solⁿ - Race

- (1) • 5000 m above
• your car: 400 m/s
• plane: 3000 m/s
• plane ahead by 40 m

What is rate of change of distance b/n plane & car?



- (3) h = height above earth for plane
 y = distance ahead for plane
 d = distance b/n plane & car

(4) $y^2 + h^2 = d^2$

(5) $\frac{d}{dt} y^2 + \frac{d}{dt} h^2 = \frac{d}{dt} d^2$

$2y \frac{dy}{dt} + 0 = 2d \frac{dd}{dt}$ (h is constant)

(6) $y = 40$

$\frac{dy}{dt} = 3000 - 400 = 2600$ (Car is also moving)

$h = 5000$

$d^2 = h^2 + y^2 = 5000^2 + 40^2 = 25000000 + 1600$

$d = \sqrt{25001600}$

$\therefore$

$2 \cdot 40 \cdot 2600 + 0 = 2 \cdot \sqrt{25001600} \cdot \frac{dd}{dt}$

(7) $\therefore \frac{dd}{dt} = \frac{2 \cdot 40 \cdot 2600}{2 \sqrt{25001600}} = \boxed{\frac{104000 \text{ km/s}}{\sqrt{25001600}}}$

$= \frac{104000}{40 \sqrt{15626}}$

$= 2600 / \sqrt{15626}$

Question 7 - Soluⁿ

$$\text{Let } a = 1$$

$$\text{Then } L(x, a) = f(a) + f'(a)(x - a)$$

$$f(x) = \operatorname{arccot}(x)$$

$$f'(x) = \frac{-1}{1+x^2}$$

$\therefore$

$$L(x, a) = \operatorname{arccot}(a) - \frac{(x-a)}{1+a^2}$$

$$L(x, 1) = \operatorname{arccot}(1) - \frac{(x-1)}{1+1^2}$$

$$= \frac{\pi}{4} - \frac{x-1}{2}$$

$$\therefore L\left(\frac{8}{9}, 1\right) = \frac{\pi}{4} - \frac{\frac{8}{9}-1}{2}$$

$$= \frac{\pi}{4} - \frac{-\frac{1}{9}}{2}$$

$$= \frac{\pi}{4} + \frac{1}{18}$$

Question 8 - Solution

$$f(x) = x^3 - 8x^2 + 16x = x(x^2 - 8x + 16) = x(x-4)^2$$

$$f'(x) = 3x^2 - 16x + 16 = (3x-4)(x-4)$$

$$f''(x) = 6x - 16 = 2(3x-8)$$

① f^h crosses x -axis at 0 & 4 since $f(x) = 0$ at 0 & 4.

② Crit points: when $f'(x) = 0$ or $f(x)$ DNE.

$\therefore$ since $f'(x)$ always exists, we only need to check when $f'(x) = 0$

$$f'(x) = 0 \text{ when } (x-4) = 0 \Rightarrow x = 4$$

$$(3x-4) = 0 \Rightarrow x = 4/3$$

Inf points: when $f''(x) = 0$

$$\therefore \text{ when } (3x-8) = 0 \Rightarrow x = 8/3$$

3/4

	$x < 4/3$	$4/3 < x < 8/3$	$8/3 < x < 4$	$4 < x$
$(x-4)$	-	-	-	+
$(3x-4)$	-	+	+	+
$(3x-8)$	-	-	+	+
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
Slope	/	\	\	/
Concave	∩	∩	∪	∪

$\therefore$ graph:

