Homework 9

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An exercise marked with the symbol $\star$ is considered more difficult and will not be an exam question.

Exercise 1 For the following:

- Find the intervals on which f is increasing/decreasing.
- Find the local max/min values of f.
- Find the intervals of concavity and the inflection points.

(1) $f(x) = 2x^3 - 9x^2 + 12x - 3$

(2) $f(x) = \frac{x}{x^2+1}$

(3) $f(x) = \cos^2(x) - 2\sin(x)$, $0 \leq x \leq 2\pi$.

(4) $f(x) = x^2 \ln(x)$

(5) $f(x) = x^4 e^{-x}$

Solution. (1)
- Increasing on $(-\infty, 1)$ and $(2, \infty)$. Decreasing on $(1, 2)$
- Local max at $f(1) = 2$ and local min at $f(2) = 1$
- Concave up on $(3/2, \infty)$ and concave down on $(-\infty, 3/2)$. Inflection point at $(3/2, 3/2)$.

(2)
- Increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$
- Local max at $f(1) = 1/2$ and local min at $f(-1) = -1/2$.
- Concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ and concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$. Inflection points are at $(-\sqrt{3}, -\sqrt{3}/4)$, $(0, 0)$, and $(\sqrt{3}, \sqrt{3}/4)$.

(3)
- Increasing on $(\pi/2, 3\pi/2)$ and decreasing on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$.
- Local min at $f(\pi/2) = -2$ and local max at $f(3\pi/2) = 2$.
- Concave up on $(\pi/6, 5\pi/6)$ and concave down on $(0, \pi/6)$, $(5\pi/6, 3\pi/2)$, and $(3\pi/2, 2\pi)$. Inflection points at $(\pi/6, -1/4)$ and $(5\pi/6, -1/4)$.

(4)
- Increasing on $(e^{-1/2}, \infty)$ and decreasing on $(0, e^{-1/2})$.
- Local min at $\frac{-1}{2e} \approx -0.18$. No local max.
- Concave up on $(e^{-3/2}, \infty)$ and concave down on $(0, e^{-3/2})$. Inflection points at $(e^{-3/2}, \frac{-3}{2e^3})$.

(5)
- Increasing on $(0, 4)$ and decreasing on $(-\infty, 0)$ and $(4, \infty)$.
- Local min at $f(0) = 0$ and local max at $f(4) = \frac{256}{e^4}$.

- Concave up on $(-\infty, 2)$ and $(6, \infty)$ and concave down on $(2, 6)$. Inflection points at $(2, 16e^{-2})$ and $(6, 1296e^{-6})$.

□

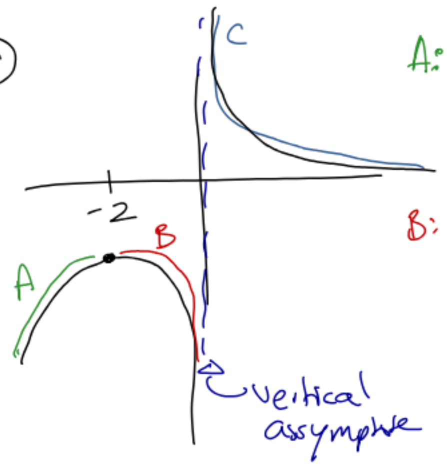
Exercise 2 Sketch the graph of a function that satisfies all the conditions.

(1)
- Vertical assymptote at $x = 0$.
- $f'(x) > 0$ if $x < -2$ and $f'(x) < 0$ if $x > -2$.
- $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$.

(2)
- $f'(0) = f'(4) = 0$
- $f'(x) = 1$ if $x < -1$
- $f'(x) > 0$ if $0 < x < 2$
- $f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4$
- $\lim_{x\to 2^-} f(x) = \infty$ and $\lim_{x\to 2^+} f(x) = \infty$.
- $f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4$
- $f''(x) < 0$ if $x > 4$.

Solution.

(1)



A: $(-\infty, -2)$: $f'(x) > 0$
$f''(x) < 0$
$\therefore$ slope is up /
concave down ∩

B: $(-2, 0)$: $f'(x) < 0$
$f''(x) < 0$
$\therefore$ slope is down \
concave down ∩

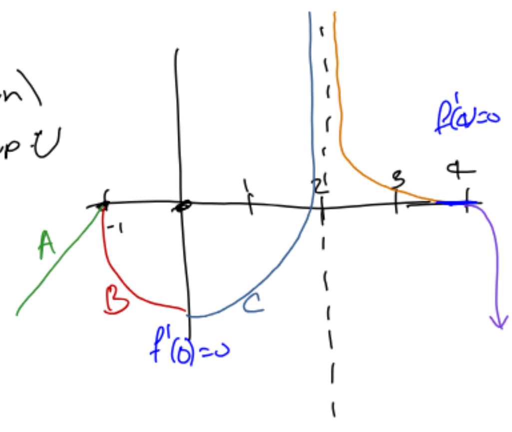
C: $(0, \infty)$: $f'(x) < 0$
$f''(x) > 0$
$\therefore$ slope is down \
Concave up ∪

(2) Intervals:

A: $(-\infty, -1)$ $f'(x) = 1$ $\therefore$ slope $= 1$

B: $(-1, 0)$ $f'(x) < 0$ $\therefore$ slope down \
$f''(x) > 0$ $\therefore$ concave up ∪

C: $(0, 2)$ $f'(x) > 0$ $\therefore$ slope up
$f''(x) > 0$ $\therefore$ concave up
$\lim_{x \to 2^-} f(x) = \infty$



D: $(2, 4)$ $f'(x) < 0$ $\therefore$ slope down
$f''(x) > 0$ $\therefore$ concave up
$\lim_{x \to 2^+} f(x) = \infty$

E: $(4, \infty)$ $f'(x) < 0$ $\therefore$ slope down
$f''(x) < 0$ $\therefore$ concave down

□

Exercise 3 Find the limit using L'hôpital's rule where appropriate. If you aren't allowed to use L'hôpital's rule, explain why.

(1) $\lim_{x\to 3} \frac{x-3}{x^2-9}$

(2) $\lim_{x\to -2} \frac{x^3+8}{x+2}$

(3) $\lim_{x\to 1/2} \frac{6x^2+5x-4}{4x^2+16x-9}$

(4) $\lim_{x\to 0} \frac{\tan(3x)}{\sin(2x)}$

(5) $\lim_{x\to 0} \frac{x^2}{1-\cos(x)}$

(6) $\lim_{\theta\to \pi} \frac{1+\cos(\theta)}{1-\cos(\theta)}$

(7) $\lim_{x\to \infty} \frac{x+x^2}{1-2x^2}$

(8) $\lim_{x\to \infty} \frac{\ln(\sqrt{x})}{x^2}$

(9) $\lim_{t\to 0} \frac{8^t-5^t}{t}$

(10) $\lim_{u\to \infty} \frac{e^{u/10}}{u^3}$

(11) $\lim_{x\to 0} \frac{x-\sin(x)}{x-\tan(x)}$

(12) $\lim_{x\to \infty} \frac{(\ln(x))^2}{x}$

(13) $\lim_{x\to 0} \frac{\cos(mx)-\cos(nx)}{x^2}$

(14) $\lim_{x\to 1} \frac{x\sin(x-1)}{2x^2-x-1}$

(15) $\star\lim_{x\to 0^+} \frac{x^x-1}{\ln(x)+x-1}$

(16) $\lim_{x\to 0} \frac{e^x-e^{-x}-2x}{x-\sin(x)}$

(17) $\lim_{x\to \infty} \sqrt{x}e^{-x/2}$

(18) $\lim_{x\to -\infty} x\ln\left(1-\frac{1}{x}\right)$

(19) $\lim_{x\to \infty} x^{3/2}\sin(1/x)$

(20) $\lim_{x\to ((\pi/2)^-)} \cos(x)\sec(5x)$

(21) $\lim_{x\to 0} (\csc(x)-\cot(x))$

(22) $\lim_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{\arctan(x)}\right)$

(23) $\lim_{x\to 1^+} \left(\ln(x^7 - 1) - \ln(x^5 - 1)\right)$

(24) $\star\lim_{x\to 0^+} (\tan(2x))^x$

(25) $\lim_{x\to\infty} x^{(\ln(2))/(1+\ln(x))}$

(26) $\lim_{x\to\infty} x^{e^{-x}}$

(27) $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$

(28) $\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$

Solution. (1) $\frac{1}{6}$

(2) 12

(3) $\frac{11}{20}$

(4) 3/2

(5) 2

(6) 0 - Can't use L'Hôpital's rule!

(7) $-1/2$

(8) 0

(9) $\ln(8/5)$

(10) ∞

(11) $-1/2$

(12) 0

(13) $\frac{1}{2}(n^2 - m^2)$

(14) 1/3

(15) 0 - Can't use L'Hôpital's rule!

(16) 2

(17) 0

(18) -1

(19) 1

(20) 1/5

(21) 0

(22) 0

(23) $\ln(7/5)$

(24) 1

(25) 2

(26) 1

(27) $e^{2/\pi}$

(28) e^{-8}

□

Exercise 4 Find two numbers whose difference is 100 and whose product is a minimum.
Solution. Since the difference is 100 we can let the two numbers be x and $x+100$. We want to minimize $f(x) = (x+100)x = x^2 + 100x$. There is an absolute min at $x = -50$ therefore our numbers are 50 and -50 □

Exercise 5 The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?
Solution. Since the sum is 16 we know x and $16 - x$ are our two numbers. We want to minimize $S(x) = x^2+(16-x)^2$. This happens when $x = 8$ and therefore $y = 8$ as well. Therefore, the smallest value is $8^2 + 8^2 = 128$. □

Exercise 6 Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.
Solution. Let x and y be the sides of the rectangle. Therefore $xy = 1000$ and $P = 2x + 2y$. Then $y = 1000/x$ and plugging in we get $P(x) = 2x + x/500$. Minimizing for P we find $x = \sqrt{1000}$ Therefore $y = \sqrt{1000}$ and the smallest perimeter is $4\sqrt{1000}$. □

Exercise 7 A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs 10 per square meter. Material for the sides cost 6 per square meter. Find the cost of materials for the cheapest such container.
Solution. Let $V = \ell wh$ be the volume of the box. Since $\ell = 2w$ we have $V = (2w)(w)h$ and as $V = 10$ we have $10 = 2w^2h$ implies $h = 5/w^2$. The cost is $C(w) = 10(2w \cdot w) + 6(2(2wh) + 2(hw)) = 20w^2 + 36wh = 20w^2 + 180/w$. Minimizing for cost we find $w = \sqrt[3]{9/2}$ gives us minimum cost which is equal to roughly \$163.54. □