Homework 7

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An exercise marked with the symbol $\star$ is considered more difficult and will not be an exam question.

Questions from this week are harder than normal because they are almost all word problems. It is recommended to use the book for assistance if you get stuck. The book is available in the library free of charge if you didn't buy the book. I've noted which question each question comes from in version 8 of the book to help. §3.7.1 means that it is Question 1 from section 3.7 in the book.

Exercise 1 (§3.7.2) A particle moves according to a law of motion of $f(t) = \frac{9t}{t^2+9}$ where t is in seconds and $f(t)$ in feet. **Hint:**Recall that velocity is a change of position and acceleration is a change of velocity.

(1) Find the velocity at time t.

(2) What is the velocity after 1 second?

(3) When is the particle at rest?

(4) When is the particle moving in the positive direction?

(5) Find the total distance traveled during the first 6 seconds.

(6) Find the acceleration at time t and after 1 second.

Solution. (1) Velocity is the derivative of position. THerefore $v(t) = f'(t) = \frac{-9(t^2-9)}{(t^2+9)^2}$ and is measured in ft/s.

(2) $v(1) = 0.72$ ft/s

(3) The particle is at rest when $v(t) = 0$. in other words, when $t^2 - 9 = 0$. This is true when $t = \pm 3$ and since t is always positive, $t = 3$.

(4) The particle is moving in the positive dir whenever $v(t) > 0$. This is true whenever $\frac{-9(t^2-9)}{(t^2+9)^2} > 0$. Since $t^2 + 9$ is always positive, we only need to look at numerator. We need:

$$\begin{aligned} -9(t^2-9) &> 0 \\ t^2 - 9 &< 0 \\ t^2 &< 9 \\ 0 \leq t &< 3 \end{aligned}$$

(5) Since the particle changes directions at $t =$, the total distance is: $|f(6) - f(3)|$ added with $|f(3) - f(0)|$.

$$|f(6) - f(3)| = 3/10 \qquad |f(3) - f(0)| = 3/2$$

(distance is always positive). Therefore, total distance is $9/5 = 1.8$ ft.

(6) Acceleration is the derivative of velocity. So

$$a(t) = v'(t) = \frac{18t(t^2 - 27)}{(t^2 + 9)^3} ft/s^2$$

So

$$a(1) = -0.468 ft/s^2$$

□

Exercise 2 (§3.7.8) If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t secons is $h(t) = 80t - 16t^2$.

(1) What is the maximum height reached by the ball?

(2) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

Solution. (1) Velocity: $v(t) = h'(t) = 80 - 32t$. We want when $v(t) = 0$ as that's when the ball is at max height. $v(t) = 0$ when $80 = 32t$. In other words, when $t = 5/2$. Then $h(5/2) = 100$ ft giving us the answer.

(2) First we calculate when we hit 96: $h(t) = 96$ when $t = 3$ and when $t = 2$. Therefore the velocities are $v(2) = 16 ft/s$ on the way up and $v(3) = -16 ft/s$ on the way down.

□

Exercise 3 (§3.7.14) A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of $60 cm/s$. Find the rate at which the area within the circle is increasing after

(1) 1s

(2) 3s

(3) 5s

Solution. Recall that area is $A(t) = \pi r^2$ and we are told that $r(t) = 60t$.

(1) $7200\pi\ cm^2/s$

(2) $21,600\pi\ cm^2/s$

(3) $36,000\pi\ cm^2/s$

□

Exercise 4 (§3.7.18) If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000(1 - \frac{1}{40}t)^2$$

Find the rate at which water is draining from the tank after:

(1) 5 min

(2) 10 min

(3) 20 min

(4) 40 min

Solution. $V'(t) = -250\left(1 - \frac{1}{40}t\right)$

(1) -218.75 gal/min

(2) -187.5 gal/min

(3) -125 gal/min

(4) 0 gal/min

□

Exercise 5 (§3.7.20) Newton's law of gravitation says that the maginitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

(1) Find dF/dr.

(2) Explain the meaning of dF/dr.

(3) What does the negative sign mean?

(4) Suppose it is known that the earth attracts an object with a force that decreases at the rate of $2N/km$ when $r = 20,000$km. How fast does this force change when $r = 10,000km$?

Solution. (1) $F' = -2(GmM)r^{-3}$

(2) It is the rate of change of the force with respect to the distance between the bodies.

(3) THe minus sign indicates that as the distance r increases, the maginitude of the force decreases between the two bodies.

(4) If $F'(20,000) = -2$ then:

$$-2 = \frac{-2GmM}{20,000^3}$$

Therefore $GmM = 20,000^3$

$$F'(10,000) = \frac{-2GmM}{10,000^3} = -16N/km$$

□

Exercise 6 (§3.7.26) The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$, the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b.
Solution. $a = 140$ and $b = 6$. □

Exercise 7 (§3.7.34) If R denotes the reaction fo the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula:

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate unites of brightness. Find the sensitivity.

$S = \frac{dR}{dx} = \frac{-54.4x^{-0.6}}{(1+4x^{0.4})^2}$

Exercise 8 (§3.8.4) A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and $25,600$ after 6 hours.
Hint: use $y(t) = y(0)e^{kt}$.

(1) What is the relative growth rate? (What is the value of k?)

(2) What was the initial size of the culture?

(3) Find an expression for the number of bacteria after t hours.

(4) Find the number of cells after 4.5 hours.

(5) Find the rate of growth after 4.5 hours.

(6) When will the population reach $50,000$?

Solution. (1) $k = \frac{3}{2}\ln(2)$

(2) $y(0) = 50$

(3) $y(t) = 50e^{(3/2)(\ln(2))t} = 50(2)^{1.5t}$

(4) $y(4.5) \approx 5382$ bacteria.

(5) $\frac{dy}{dt} = k \cdot y(t)$. Therefore $y'(4.5) \approx 5596$ bacteria/h

(6) $y(t) = 50,000$ when $t \approx 6.64$h

□

Exercise 9 (§3.8.10) A sample of tritium-3 decayed to 94.5% of its original amount after a year.

(1) What is the half-life of tritium-3?

(2) How long would it take the sample to decay to 20% of its original amount?

Solution. Recall $y(t) = y(0)e^{kt}$

(1) $y(1) = 0.945y(0)$, implying $k = \ln(0.945)$. Half-life is when $y(t) = \frac{1}{2}y(0)$. This gives us $t \approx 12.25$ years.

(2) We need to find $y(t) = 0.2y(0)$. This happens when $t \approx 28.45$ years.

□

Exercise 10 (§3.8.16) In a muder investigation, the temperature of the corpse was $32.5°C$ at $13:30$ and $30.3°C$ an hour later. Normal body temperature is $37.0°C$ and the temperature fo the surroundings was $20.0°C$. When did the murder take place?

Solution. Recall that Newton's law of cooling says:

$$\frac{dT}{dt} = k(T - T_s)$$

where T_s is surrounding temperature. So we know $T(0) = 32.5$ and $T(1) = 30.3$ from the problem.

Let $y = T - T_s = T - 20$. Then $y(0) = T(0) - 20 = 32.5 - 20 = 12.5$ and $y(1) = 10.3$. Recall that $\frac{dT}{dt} = y'(t) = ky$ and $y(t) = y(0)e^{kt}$. Since $y(1) = 10.3 = 12.5e^k$ we know $k = \ln\left(\frac{10.3}{22.5}\right)$.

To calculate when the murder took place, we find when $y(t) = 37 - 20 = 17$. In other words, when $t \approx -1.588$ h, which is roughly 95 minutes before $13:30$. So the murder took place at $11:55$! □

Exercise 11 (§3.9.6) The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80mm?

Solution. The volumne of a sphere is $V = \frac{4}{3}\pi r^3$. Take the derivative and solve at $r = 80$ to get: $25,600\pi\ mm^3/s$. □

Exercise 12 (§3.9.14) If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Solution. If the surface area is $S = 4\pi r^2$, then the rate of decrease means $\frac{dS}{dt} = -1$. We also know $2r = d$ where d is the diameter (since the question is about diameter). Therefore:

$$-1 = \frac{dS}{dt} = \frac{d}{dt}4\pi\left(\frac{d}{2}\right)^2 = 2\pi d\frac{dd}{dt}$$

And therefore $\frac{dd}{dt} = \frac{-1}{2\pi d}$. At $d = 10$ we have $\frac{dd}{dt} = -\frac{1}{20\pi}$ giving us the rate of decrease.

□

Exercise 13 (§3.9.18) A spotlight on the ground shines on a wall 12 m away. If a 2 m tall person walks from teh spotlight toward the building at a speed of 1.6 m/s, how fast ist he length of his shadow on the building decreasing when he is 4 m from the building?
Solution. This is best shown through a triangle. The solution is -0.6 m/s. □

Exercise 14 (§3.9.22) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on they dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?
Solution. This is best shown through a triangle. The solution is $\frac{\sqrt{65}}{8} \approx 1.01$ m/s. □

Exercise 15 (§3.9.30) A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At which rate is the angle between the string and the horizontal decreasing when 200 ftof string has been let out?
Solution. This is best shown through a triangle. THe solution is $\frac{-1}{50}$ rad/s. □

Exercise 16 (§3.9.40) Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15cm to 20cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm?
Solution. We want to find $\frac{dB}{dt}$ when $L = 18$ using our formulas.

$$\frac{dB}{dt} = \frac{dB}{dW}\frac{dW}{dL}\frac{dL}{dt} = \left(0.007 \cdot \frac{2}{3}W^{-1/3}\right)\left(0.12 \cdot 2.53 \cdot L^{1.53}\right)\left(\frac{20-15}{10,000,000}\right)$$

This gives us: $\approx 1.045 \times 10^{-8}$g/yr. □

Exercise 17 (§3.9.46) A ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?
Solution. This is best shown through a circle. The solution is 8π m/min. □

Exercise 18 (§3.9.50) The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock? **Hint:** Try using the law of cosines.
Solution. This is best shown through a circle. The distance between the tips of the hands is decreasing at a rate of 18.6 mm/h which is roughly 0.005 mm/s. □

Exercise 19 Find the linearization $L(x, a)$ of the following:

(1) $f(x) = \sin(x)$ when $a = \pi/6$

(2) $f(x) = 2^x$ when $a = 0$

Solution. (1) $L(x, \pi/6) = \frac{\sqrt{3}}{2}x + \frac{1}{2} - \frac{\pi\sqrt{3}}{12}$

(2) $L(x, 0) = 1 + \ln(2)x$

$\square$