Facial Weak Order

Aram Dermenjian

Joint work with: Christophe Hohlweg (LACIM) and Vincent Pilaud (CNRS & LIX)

Université du Québec à Montréal

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History and Background

- The weak order was introduced on Coxeter groups by Björner in 1984, it was shown to be a lattice.
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- **Finite Coxeter System** \((W, S)\) such that
  \[ W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle \]

  where \( m_{i,j} \in \mathbb{N}^* \) and \( m_{i,j} = 1 \) only if \( i = j \).
- A **Coxeter diagram** \( \Gamma_W \) for a Coxeter System \((W, S)\) has \( S \) as a vertex set and an edge labelled \( m_{i,j} \) when \( m_{i,j} > 2 \).
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Example

\[ W_{B_3} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle \]

\[ \Gamma_{B_3} : \]

\[ s_1 \quad 4 \quad s_2 \quad s_3 \]
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Let $(W, S)$ be a Coxeter system.

- Let $w \in W$ such that $w = s_1 \ldots s_n$ for some $s_i \in S$. We say that $w$ has length $n$, $\ell(w) = n$, if $n$ is minimal.

- Let the (right) weak order be the order on the Cayley graph where $w \xrightarrow{wS}$ and $\ell(w) < \ell(ws)$.

- For finite Coxeter systems, there exists a longest element in the weak order, $w_\circ$. 

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Example

Let $\Gamma_{A_2}$: $s \rightarrow t$. $sts = w_0 = tst$

For finite Coxeter systems, there exists a longest element in the weak order, $w_{\circ}$.
Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers,
  2. gave a global definition of this order combinatorially, and
  3. showed that the poset for this order is a lattice.

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Parabolic Subgroups

Let $I \subseteq S$.

- $W_I = \langle I \rangle$ is the **standard parabolic subgroup** with long element denoted $w_0, I$.
- $W^I := \{ w \in W \mid \ell(w) \leq \ell(ws), \text{ for all } s \in I \}$ is the set of minimal length coset representatives for $W/W_I$.
- Any element $w \in W$ admits a unique factorization $w = w^I \cdot w_I$ with $w^I \in W^I$ and $w_I \in W_I$.
- By convention in this talk $xW_I$ means $x \in W^I$.
- **Coxeter complex** - $\mathcal{P}_W$ - the abstract simplicial complex whose faces are all the standard parabolic cosets of $W$. 

![Diagram of Coxeter complex](image-url)
Facial Weak Order

Definition (Krob et.al. [2001], Palacios, Ronco [2006])

The (right) facial weak order is the order $\leq_F$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types:

1. $xW_I \leq xW_{I\cup\{s\}}$ if $s \notin I$ and $x \in W_{I\cup\{s\}}$,
2. $xW_I \leq xw_{I\setminus\{s\}}W_{I\setminus\{s\}}$ if $s \in I$,

where $I \subseteq S$ and $x \in W^I$. 
Facial weak order example

(1) \( xW_I \leq xW_{I \cup \{s\}} \) if \( s \notin I \) and \( x \in W_{I \cup \{s\}} \)

(2) \( xW_I \leq xw_0,w_{I \setminus \{s\}}W_{I \setminus \{s\}} \) if \( s \in I \)

![Diagram of facial weak order example]
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Root System

- Let \((V, \langle \cdot, \cdot \rangle)\) be a Euclidean space.
- Let \(W\) be a group generated by a set of reflections \(S\). \(W \hookrightarrow O(V)\) gives representation as a finite reflection group.
- The reflection associated to \(\alpha \in V \setminus \{0\}\) is
  \[s_\alpha(v) = v - \frac{2 \langle v, \alpha \rangle}{||\alpha||^2} \alpha \quad (v \in V)\]

- A root system is \(\Phi := \{\alpha \in V \mid s_\alpha \in W, ||\alpha|| = 1\}\)
- We have \(\Phi = \Phi^+ \sqcup \Phi^-\) decomposable into positive and negative roots.
Inversion Sets

Let \((W, S)\) be a Coxeter system. Define \((left)\ inversion\ sets\) as the set \(N(w) := \Phi^+ \cap w(\Phi^-)\).

**Example**

Let \(\Gamma_{A_2} : s \rightarrow t\), with \(\Phi\) given by the roots:

\[
N(ts) = \Phi^+ \cap ts(\Phi^-)
\]

\[
= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}
\]

\[
= \{\alpha_t, \gamma\}
\]
Weak order and Inversion sets

Given \( w, u \in W \) then \( w \leq_R u \) if and only if \( N(w) \subseteq N(u) \).

Example

Let \( \Gamma_{A_2} : s \rightarrow t \), with \( \Phi \) given by the roots 

\[
\phi = \{ \alpha_t, \gamma \} \quad \Phi^+ = \{ \alpha_s, \gamma \} 
\]

\[
\{ \alpha_t \} \quad \{ \alpha_s \} 
\]

\[ \gamma = \alpha_s + \alpha_t \]
Root Inversion Set

Definition (Root Inversion Set)

Let $xW_I$ be a standard parabolic coset. The root inversion set is the set

$$R(xW_I) := x(\Phi^- \cup \Phi_I^+)$$

Note that $N(x) = R(xW_{\emptyset}) \cap \Phi^+$. 
Root Inversion Set

Example

\[ R(sW\{t\}) = s(\Phi^- \cup \Phi^+\{t\}) \]
\[ = s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \]
\[ = \{\alpha_s, -\gamma, -\alpha_t, \gamma\} \]
**Root Inversion Set**

**Example**

\[
R(sW_{\{t\}}) = s(\Phi^- \cup \Phi^+_{\{t\}}) \\
= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \\
= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}
\]
Equivalent definitions

**Theorem (D., Hohlweg, Pilaud [2016])**

The following conditions are equivalent for two standard parabolic cosets $xW_I$ and $yW_J$ in the Coxeter complex $\mathcal{P}_W$:

1. $xW_I \leq_F yW_J$
2. $R(xW_I) \setminus R(yW_J) \subseteq \Phi^-$ and $R(yW_J) \setminus R(xW_I) \subseteq \Phi^+$.
3. $x \leq_R y$ and $xw_{\circ,I} \leq_R yw_{\circ,J}$.

**Remark**  Note that showing (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) is easy, but (2) $\Rightarrow$ (1) is more difficult. We used induction on the symmetric difference between the root inversion sets for the proof.
Equivalence for type $A_2$ Coxeter System

\[ \alpha_s \gamma \alpha_t \]

\[ e \leq R y \]

\[ x \leq_R y \]

\[ x_{W_I} \leq_F y_{W_J} \]
Equivalence for type $A_2$ Coxeter System

$\alpha_s$, $\gamma$, $-\alpha_t$

$\alpha_t$, $-\gamma$, $-\alpha_s$

$xW_I \leq_F yW_J$

$R(xW_I) \setminus R(yW_J) \subseteq \Phi^-$

$R(yW_J) \setminus R(xW_I) \subseteq \Phi^+$

$x \leq_R y$

$xw_{0,I} \leq_R yw_{0,J}$
Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type $A$ using inversion tables. They
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  3. showed that the poset for this order is a lattice.
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Facial weak order lattice

Theorem (D., Hohlweg, Pilaud [2016])

The facial weak order \((\mathcal{P}_W, \leq_F)\) is a lattice with the meet and join of two standard parabolic cosets \(xW_I\) and \(yW_J\) given by:

\[
\begin{align*}
    xW_I \land yW_J &= z \land W_{K \land}, \\
    xW_I \lor yW_J &= z \lor W_{K \lor}.
\end{align*}
\]

where,

\[
\begin{align*}
    z \land &= x \land y & \text{and} & \quad K \land &= D_L(z^{-1}(xW_o,l \land yW_o,J)) \quad \text{and} \\
    z \lor &= xW_o,l \lor yW_o,J & \text{and} & \quad K \lor &= D_L(z^{-1}(x \lor y))
\end{align*}
\]

Corollary (D., Hohlweg, Pilaud [2016])

The weak order is a sublattice of the facial weak order lattice.
Example: $A_2$ and $B_2$
Example: $A_2$ and $B_2$

Example (Meet example)

Recall

$$xW_I \land yW_J = z_\land W_K_\land$$
where $z_\land = x \land y$

$$K_\land = D_L(z_\land^{-1}(xw_o,I \land yw_o,J))$$

We compute $ts \land stsW\{t\}$.

$$z_\land = ts \land sts = e$$

$$K_\land = D_L(z_\land^{-1}(tsw_o,\emptyset \land stsw_o,t))$$
$$= D_L(e(ts \land stst))$$
$$= D_L(ts) = \{t\}.$$
**Proof outline**

Recall that \( x W_I \leq_F y W_J \iff x \leq_R y \), and \( x w_o, I \leq_R y w_o, J \).

We want to show that \( x W_I \wedge y W_J = z \wedge W K \) where \( z = x \wedge y \) and \( K = D_L(z^{-1}(xw_o, I \wedge yw_o, J)) \).

- First we show that this element is in the Coxeter complex \( z \in W^K \).
- We then show it’s a lower bound: \( x \wedge y \leq_R x, y \). Also, \( w_o, K \leq_R z^{-1}(xw_o, I \wedge yw_o, J) \) implies \( z w_o, K \leq_R xw_o, I \wedge yw_o, J \).
- Finally we show uniqueness by supposing there exists another element \( z W_K \leq_F x W_I, y W_J \). Then we have \( z \leq_R x \wedge y = z \).

Join is found by an anti-automorphism.
Möbius function

Recall that the *Möbius function* of a poset \((P, \leq)\) is the function \(\mu : P \times P \to \mathbb{Z}\) defined inductively by

\[
\mu(p, q) := \begin{cases} 
1 & \text{if } p = q, \\
- \sum_{p \leq r < q} \mu(p, r) & \text{if } p < q, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proposition (D., Hohlweg, Pilaud [2016])**

The Möbius function of the facial weak order is given by

\[
\mu(eW_\emptyset, yW_J) = \begin{cases} 
(-1)^{|J|}, & \text{if } y = e, \\
0, & \text{otherwise.}
\end{cases}
\]
Quotients of the facial weak order
Lattice Congruences

Definition

A **lattice congruence** is an equivalence relation \( \equiv \) on a lattice \((L, \leq)\) such that for each \( x_1 \equiv x_2 \) and \( y_1 \equiv y_2 \) then

1. \( x_1 \land y_1 \equiv x_2 \land y_2 \), and
2. \( x_1 \lor y_1 \equiv x_2 \lor y_2 \).

Theorem (D., Hohlweg, Pilaud [2016])

*Given a lattice congruence \( \equiv \) on \((W, \leq_R)\), the equivalence classes on \((\mathcal{P}_W, \leq_F)\) defined by*

\[
x W_I \equiv y W_J \iff x \equiv y \text{ and } x w_{\circ, I} \equiv y w_{\circ, J}
\]

*give us a lattice congruence.*
Corollary (D., Hohlweg, Pilaud [2016])

Let the (left) root descent set of a coset $xW_I$ be the set of roots

$$D(xW_I) := R(xW_I) \cap \pm \Delta \subseteq \Phi.$$ 

Let $xW_I \equiv_{\text{des}} yW_J$ if and only if $D(xW_I) = D(yW_J)$. 

\[ \text{Diagram: Facial Boolean Lattice} \]
Corollary (D., Hohlweg, Pilaud [2016])

Let \( c \) be any Coxeter element of \( W \). Let \( \equiv^c \) be the \( c \)-Cambrian congruence (see Reading [Cambrian Lattice, 2004]). Then let

\[ xW_I \equiv^c yW_J \iff x \equiv^c y \text{ and } xw_\circ,I \equiv^c yw_\circ,J. \]
Thank you!